

Probing EFX via PMMS: (Non-)Existence Results in Discrete Fair Division

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Fair Division

Fairly divide **resources** among a group of **agents**.

Fairly divide an **inheritance** among a group of **siblings**.

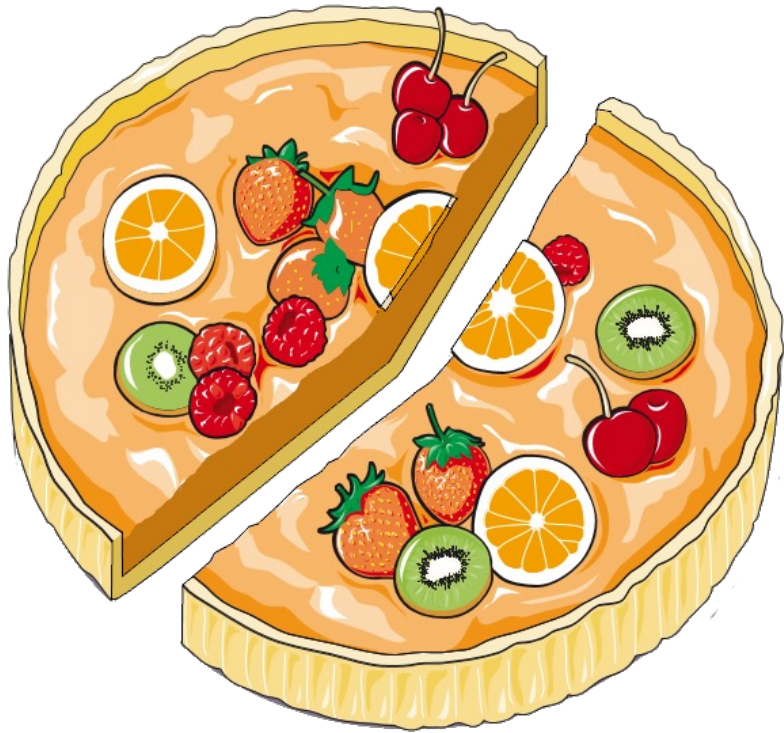


Fairly divide **tasks** among a group of **employees**.



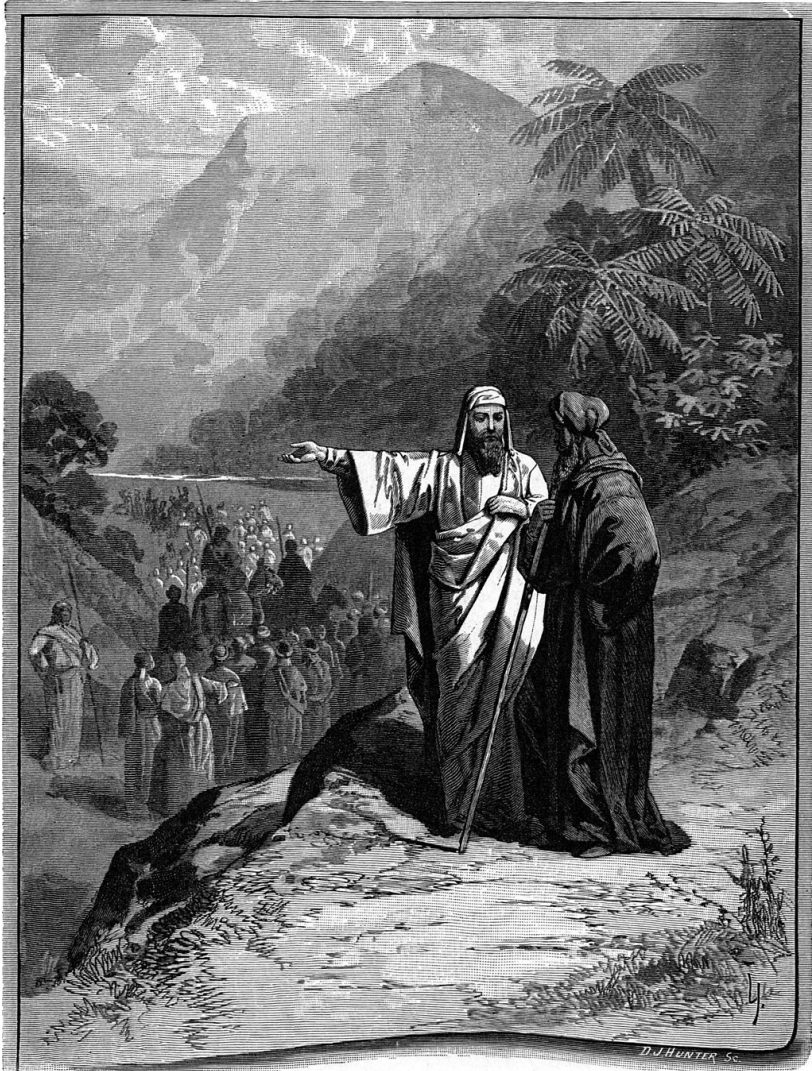
Question: What is the **strongest** fairness notion possible in the **worst case**?

Two Agents: Cut And Choose



For **two agents**, the ideal method is **cut and choose**.

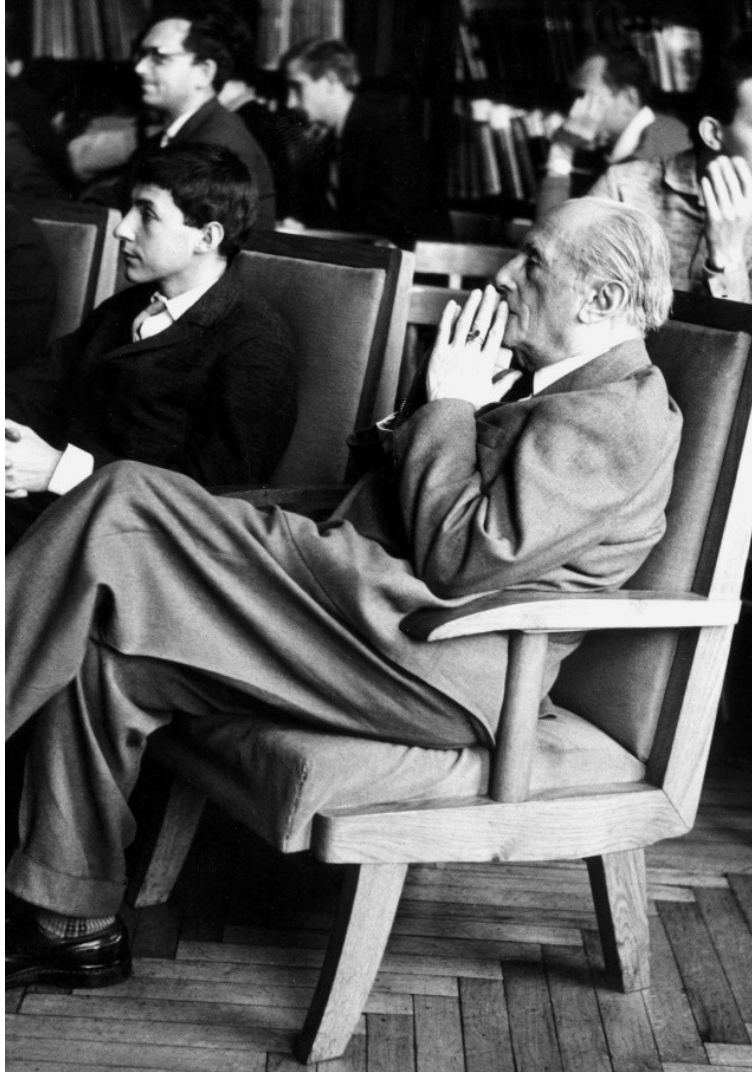
Two Agents: Cut And Choose



For **two agents**, the ideal method is **cut and choose**.

A **cuts** resources into two parts
B **chooses** the one it prefers

Two Agents: Cut And Choose



For **two agents**, the ideal method is **cut and choose**.

A **cuts** resources into two parts
B **chooses** the one it prefers

Question: How can **cut and choose** be generalized to **more agents**?

The Model

	item a	item b	item c	item d
agent 1	3	1	1	1
agent 2	2	3	1	1
agent 3	1	1	1	1

For now, assume **additive** valuations: $v_i(X_i) = \sum_{g \in X_i} v_i(\{g\})$
We also consider **monotone** valuations $v_i : 2^{[m]} \rightarrow \mathbb{R}$.

Items are **indivisible** and have to be **fully allocated**
in a **deterministic** way.

Agents have **subjective preferences** and **equal entitlements**.

Fairness Notions

Allocation is **envy-free (EF)** if $v_i(X_i) \geq v_i(X_j)$ for all i, j .

	item a	item b
agent <i>i</i>	4	3

Agent *i* does not satisfy EF ❌

	item a	item b
agent <i>i</i>	4	3

Agent *i* satisfies EF ✅

EF allocations **exist** for **divisible** items.
EF allocations might **not exist** for **indivisible** items.

Fairness Notions

Allocation is **envy-free up to some good (EF1)** if for all i, j , $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for some $g \in X_j$.

	item a	item b	item c
agent i	3	3	5

Agent i does not satisfy EF ✖

Agent i satisfies EF1 ✔

Theorem [LMMS'2001]:
EF1 allocations **exist** for **indivisible** items.

Fairness Notions

Allocation is **envy-free up to any good (EFX)**
if for all i, j , $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for **any** $g \in X_j$.

	item a	item b	item c
agent i	4	2	3

Agent i does not satisfy EFX ❌

Agent i satisfies EF1 ✅

Open Problem: Does EFX **exist** for **indivisible** items?
Theorem [CGM'2020]: EFX **exists** for **three agents**.

Fairness Notions

The **maximin share**: $\mu_i(S) = \max_{X \sqcup Y = S} \min \{v_i(X), v_i(Y)\}$.

	a	b	c	d
<i>i</i>	2	4	5	6

The maximin share is 8.

Allocation is **pairwise-maximin-share-fair (PMMS)**
if for all i, j , $v_i(X_i) \geq \mu_i(X_i \cup X_j)$.

	a	b	c	d
<i>i</i>	2	4	5	6

Agent *i* does not satisfy PMMS ✖

Agent *i* satisfies EFX ✔

	a	b	c	d
<i>i</i>	2	4	5	6

Agent *i* satisfies PMMS ✔

Agent *i* satisfies EFX ✔

Fairness Notions

PMMS: $v_i(X_i) \geq \mu_i(X_i \cup X_j)$
where $\mu_i(S) = \max_{X \cup Y = S} \min \{v_i(X), v_i(Y)\}$

	item a	item b	item c	item d
agent 1	3	1	1	1
agent 2	2	3	1	1
agent 3	1	1	1	1

Comparison-based fairness (**this talk**):

For each pair of agents i and j , the condition depends on X_i and X_j .

Share-based fairness (**not in this talk**):

For each agent i , the condition depends on X_i and number of agents.

Fairness Notions

Question: What is the **strongest** fairness notion possible in the **worst case**?

	definition	status
EF	$v_i(X_i) \geq v_i(X_j)$	does not exist
PMMS	$v_i(X_i) \geq \mu_i(X_i \cup X_j)$	exists for 2 agents open for 3 agents
EFX	$v_i(X_i) \geq v_i(X_j \setminus \{g\}) \forall g \in X_j$	exists for 3 agents open for 4 agents
EF1	$v_i(X_i) \geq v_i(X_j \setminus \{g\}) \exists g \in X_j$	exists

Multiplicative approximations of fairness (**not in this talk**):
0.781-PMMS exists [K'2017] and **0.618-EFX** exists [AMN'2020].

EFX vs. PMMS

EFX and PMMS were both introduced in 2016.

The Unreasonable Fairness of Maximum Nash Welfare

IOANNIS CARAGIANNIS, University of Patras

DAVID KUROKAWA, Carnegie Mellon University

HERVÉ MOULIN, University of Glasgow and Higher School of Economics, St Petersburg

ARIEL D. PROCACCIA, Carnegie Mellon University

NISARG SHAH, Carnegie Mellon University

JUNXING WANG, Carnegie Mellon University

The *maximum Nash welfare (MNW)* solution — which selects an allocation that maximizes the product of utilities — is known to provide outstanding fairness guarantees when allocating divisible goods. And while it seems to lose its luster when applied to indivisible goods, we show that, in fact, the MNW solution is unexpectedly, strikingly fair even in that setting. In particular, we prove that it selects allocations that are envy free up to one good — a compelling notion that is quite elusive when coupled with economic efficiency. We also establish that the MNW solution provides a good approximation to another popular (yet possibly infeasible) fairness property, the maximin share guarantee, in theory and — even more so — in practice. While finding the MNW solution is computationally hard, we develop a nontrivial implementation, and demonstrate that it scales well on real data. These results lead us to believe that MNW is the ultimate solution for allocating indivisible goods, and underlie its deployment on a popular fair division website.

CCS Concepts: •**Theory of computation**→ **Algorithmic mechanism design**; •**Applied computing**→ **Economics**;

Additional Key Words and Phrases: Fair division, Resource allocation, Nash welfare

EFX vs. PMMS

EFX got a lot more attention since then.

EFX: 21,500 results

Google Scholar

"EFX" OR "envy-freeness up to any good"

Articles

About 21,500 results (0.18 sec)

Any time

Since 2025

Since 2024

Since 2021

Custom range...

Sort by relevance

Sort by date

Any type

Review articles

☐ include patents

☒ include citations

Create alert

Computing Envy-Free up to Any Good (EFX) Allocations via Local Search

[S Brânzei](#) - arXiv preprint arXiv:2510.05429, 2025 - arxiv.org

... **EFX** (envy-free up to any good) allocations of m indivisible goods among n agents with additive valuations. **EFX** ... Our algorithm employs simulated annealing with the total number of **EFX** ...

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EFX exists for three agents

[BR Chaudhury, J Garg, K Mehlhorn](#) - Journal of the ACM, 2024 - dl.acm.org

... **envy-freeness up to any good (EFX)**. Despite significant efforts by many researchers for several years, the existence of **EFX** ... In this article, we show constructively that an **EFX** allocation ...

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Improving envy freeness up to any good guarantees through rainbow cycle number

[BR Chaudhury, J Garg, K Mehlhorn](#)... - Mathematics of ..., 2024 - pubsonline.informs.org

... In particular, it is known that 0.618-**EFX** allocations exist and that **EFX** ... ϵ) -**EFX** allocation with sublinear number of unallocated goods and high Nash welfare. For this, we reduce the **EFX** ...

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PMMS: 38 results

Google Scholar

"PMMS" or "pairwise maximin share"

Articles

About 38 results (0.08 sec)

Any time

Since 2025

Since 2024

Since 2021

Custom range...

Sort by relevance

Sort by date

Any type

Review articles

☐ include patents

☒ include citations

Create alert

Exact and approximation algorithms for PMMS under identical constraints

[S Dai, G Gao, X Guo, Y Zhang](#) - ... Conference on Theory and Applications of ..., 2022 - Springer

... Nash Social Welfare allocation is always **PMMS** and EFX if the valuation ... **pairwise maximin share** allocations for identical variants. **PMMS** graph is introduced to help us find the **PMMS** ...

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Groupwise maximin fair allocation of indivisible goods

[S Barman, A Biswas, S Krishnamurthy](#)... - Proceedings of the AAAI ..., 2018 - ojs.aaai.org

... **pairwise maximin share** guarantee (**PMMS**), a notion defined by (Caragiannis et al. 2016). In **PMMS**, ... that GMMS is a strict generalization of **PMMS** and MMS. The relevance of GMMS is ...

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Exact and approximate maximin share allocations in multi-graphs

[G Christodoulou, S Mastrakoulis](#) - arXiv preprint arXiv:2506.20317, 2025 - arxiv.org

... **Pairwise maximin share** and α -**PMMS** Another fairness notion which we will consider in this work, is the **pairwise maximin share (PMMS)** ... if we can guarantee exact **PMMS** value for all ...

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About PMMS

PMMS: $v_i(X_i) \geq \mu_i(X_i \cup X_j)$
where $\mu_i(S) = \max_{X \sqcup Y = S} \min \{v_i(X), v_i(Y)\}$

1. **PMMS** is exactly the guarantee the **cutter** gets in **cut and choose**.
2. **PMMS** extends to **chores** and **mixed manna**, unlike **EFX**.
3. **PMMS** is **stronger** than **EFX** in **non-degenerate** instances.
4. **PMMS** is **computationally harder** than **EFX**.

About PMMS

1. **PMMS** is exactly the guarantee the **cutter** gets in **cut and choose**.

PMMS: $v_i(X_i) \geq \mu_i(X_i \cup X_j)$
where $\mu_i(S) = \max_{X \sqcup Y = S} \min \{v_i(X), v_i(Y)\}$

	a	b	c	d
<i>i</i>	2	4	5	6

	a	b	c	d
<i>i</i>	②	④	⑤	⑥

	a	b	c	d
<i>i</i>	②	④	⑤	⑥

Cutter cuts into two parts.
Guarantees **PMMS**.

Chooser chooses one.
Guarantees **EF**.

About PMMS

2. PMMS extends cleanly to **chores** and **mixed manna**, unlike EFX.

PMMS for **goods**, **chores**, and **mixed manna**: $v_i(X_i) \geq \mu_i(X_i \cup X_j)$
where $\mu_i(S) = \max_{X \sqcup Y = S} \min \{v_i(X), v_i(Y)\}$

	a	b	c	d
<i>i</i>	2	4	5	6

Agent *i* satisfies PMMS ✓
MMS share is 8.

	a	b	c	d
<i>i</i>	-2	-4	-5	-6

Agent *i* satisfies PMMS ✓
MMS share is -9.

EFX for **goods**: $v_i(X_i) \geq v_i(X_j \setminus \{g\}) \forall g \in X_j$

EFX for **chores**: $v_i(X_i \setminus \{g\}) \geq v_i(X_j) \forall g \in X_i$

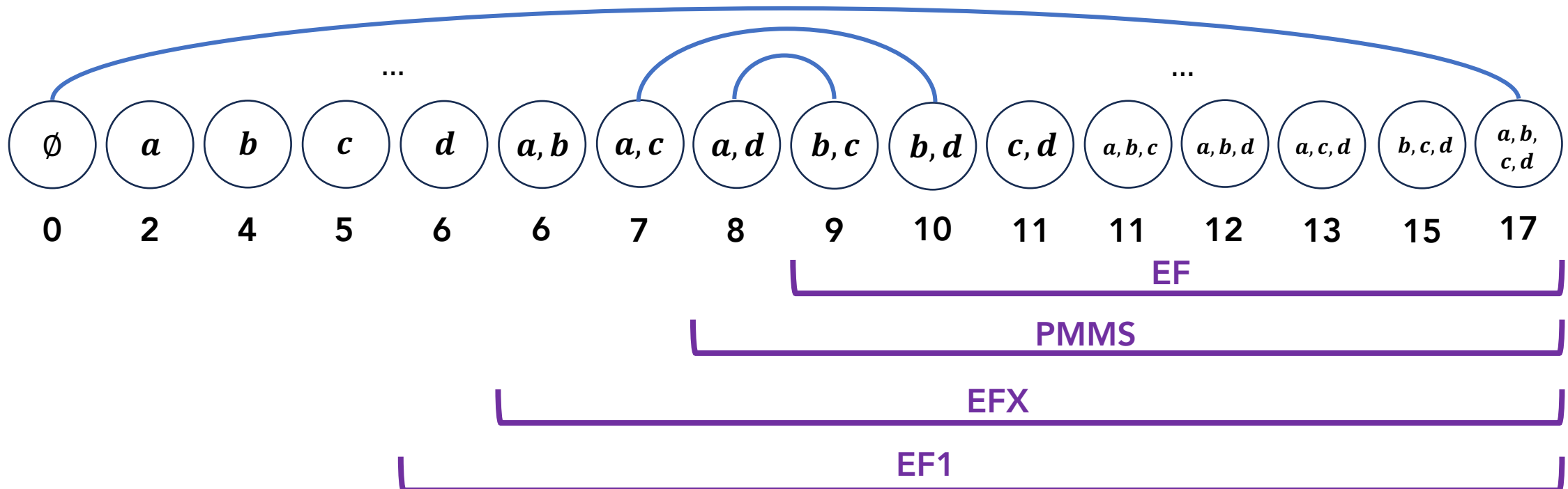
About PMMS

3. PMMS is stronger than EFX in non-degenerate instances.

	a	b	c	d
i	2	4	5	6

EF
PMMS
EFX
EF1

$v_i(X_i) \geq v_i(X_j)$
$v_i(X_i) \geq \mu_i(X_i \cup X_j)$
$v_i(X_i) \geq v_i(X_j \setminus \{g\}) \forall g \in X_j$
$v_i(X_i) \geq v_i(X_j \setminus \{g\}) \exists g \in X_j$



About PMMS

4. PMMS is computationally harder than EFX.

Theorem: Finding a PMMS allocation for two identical additive agents is NP-hard.

Proof: Reduction from the Partition problem.

Theorem [GHH'2023]:
Finding an EFX allocation for two additive agents is in P.

Research Direction

Big Open Problems:

Does **EFX** exist for **additive valuations**?
Does **PMMS** exist for **additive valuations**?

Our Goals:

Show **impossibility** for **weaker** assumptions.
Show **possibility** for **stronger** assumptions.

Negative Results

Theorem (**Negative**): For any n , there is an n -agent instance with n **monotone** valuations that admits no **PMMS** allocation.

Extending the 2-agent impossibility to n agents is non-trivial.

Theorem (**Negative**): There is an instance with **2 monotone** and **1 additive** valuation that admits no **PMMS** allocation.

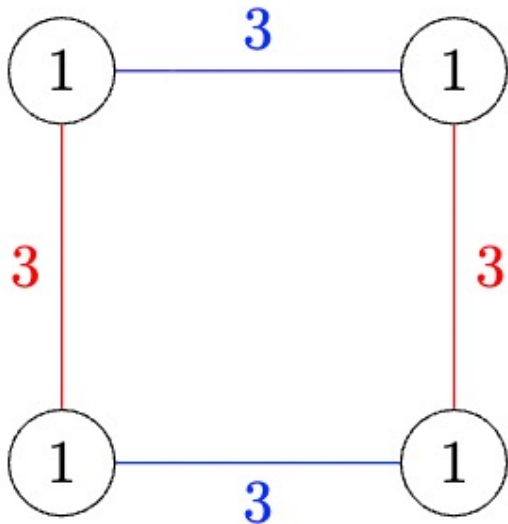
EFX known to exist for **2 monotone** and **1 additive** valuations.
Thus, **PMMS** for 3 agents requires new techniques.

Monotone Valuations

Theorem (**Negative**): For any n , there is an n -agent instance with n **monotone** valuations that admits no **PMMS** allocation.

PMMS: $v_i(X_i) \geq \mu_i(X_i \cup X_j)$ where $\mu_i(S) = \max_{X \cup Y = S} \min \{v_i(X), v_i(Y)\}$

Two agents (**red** and **blue**)



Nodes show values for **single** items.
Edges show values for **pairs** of items.

The MMS shares are $\mu_1 = \mu_2 = 3$.

No allocation satisfies
 $v_1(X_1) \geq 3$ and $v_2(X_2) \geq 3$.

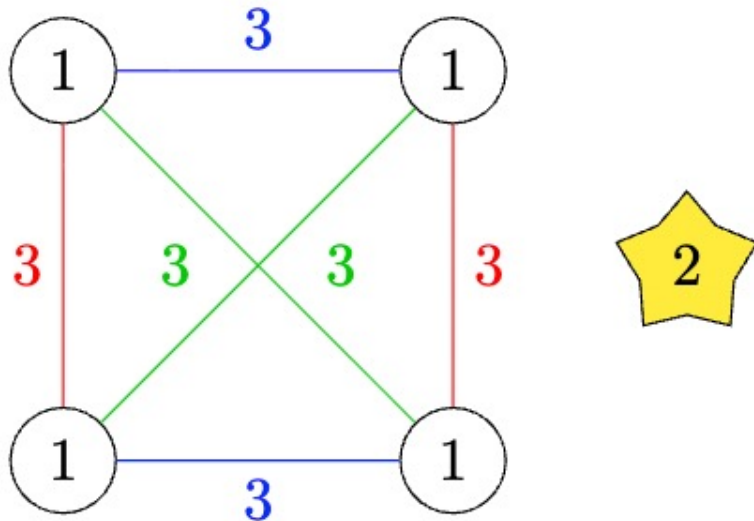
How to extend this to **3 agents**?

Monotone Valuations

Theorem (**Negative**): For any n , there is an n -agent instance with n **monotone** valuations that admits no **PMMS** allocation.

PMMS: $v_i(X_i) \geq \mu_i(X_i \cup X_j)$ where $\mu_i(S) = \max_{X \cup Y = S} \min \{v_i(X), v_i(Y)\}$

Three agents (**red**, **blue**, **green**)



Nodes show values for **single** items.
Edges show values for **pairs** of items.

One agent gets only the **star**.
The remaining items **cannot** be divided
between the other two agents.

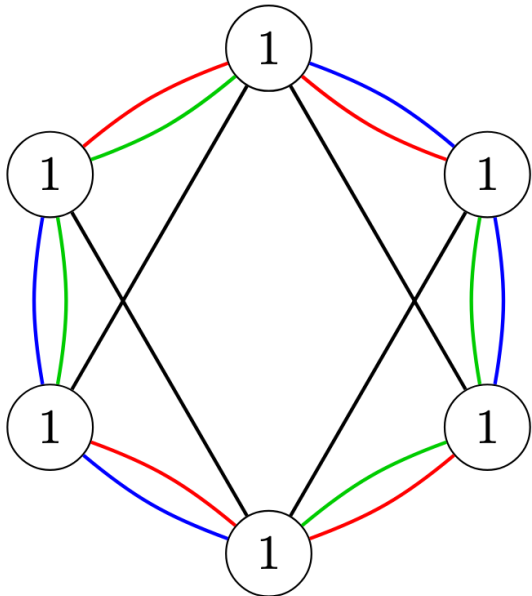
How to extend this to **4 agents**?

Monotone Valuations

Theorem (**Negative**): For any n , there is an n -agent instance with n **monotone** valuations that admits no **PMMS** allocation.

PMMS: $v_i(X_i) \geq \mu_i(X_i \cup X_j)$ where $\mu_i(S) = \max_{X \cup Y = S} \min \{v_i(X), v_i(Y)\}$

Four agents
(red, blue, green, black)



Nodes show values for **single** items.
Edges show values for **triplets** of items.

For n agents, our construction
uses $(n - 2) + O(\log n)$ items.

Open Problem:
Could $(n - 2) + O(1)$ be enough?

MMS-Feasible Valuations

Theorem (**Negative**): For any n , there is an n -agent instance with n **monotone** valuations that admits no **PMMS** allocation.

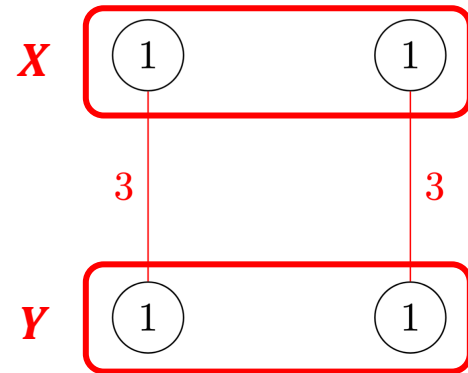
Question: Can extra assumptions on valuations help?
We consider **MMS-feasible** vals, a generalization of **additive**.

Open Problem: Does **PMMS** exist for **MMS-feasible** vals?
Open Problem: Does **PMMS** exist for **additive** valuations?

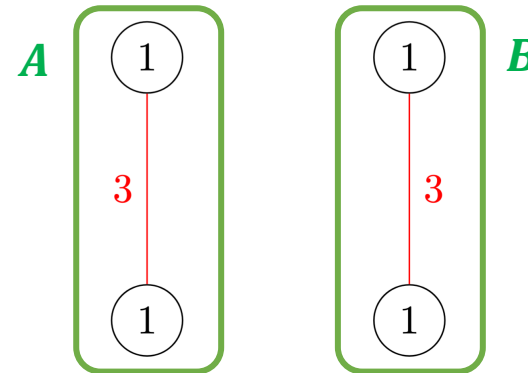
MMS-Feasible Valuations

Valuation is **MMS-feasible** if $\max\{v_i(X), v_i(Y)\} \geq \min\{v_i(A), v_i(B)\}$ for any $S \subseteq [m]$ and any two partitions $X \uplus Y = S$ and $A \uplus B = S$

Observation 1: Our earlier construction **violates MMS-feasibility**.



$$\max\{v_i(X), v_i(Y)\} = 1$$



$$\min\{v_i(A), v_i(B)\} = 3$$

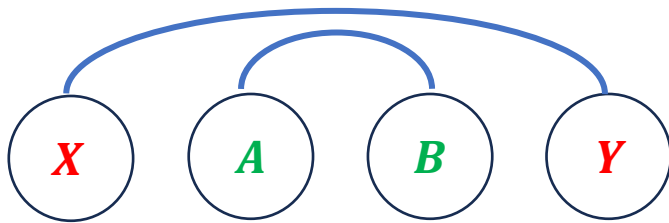
Observation 2: Additive valuations **satisfy MMS-feasibility**.

Proof: $\max\{v_i(X), v_i(Y)\} \geq (1/2) \cdot v_i(S) \geq \min\{v_i(A), v_i(B)\}$

MMS-Feasible Valuations

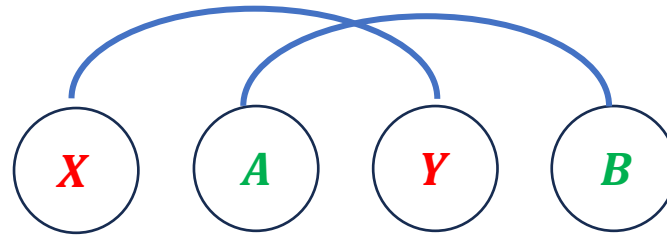
Valuation is **MMS-feasible** if $\max\{v_i(X), v_i(Y)\} \geq \min\{v_i(A), v_i(B)\}$ for any $S \subseteq [m]$ and any two partitions $X \uplus Y = S$ and $A \uplus B = S$

Recall: We order bundles from least to most valuable and draw arcs between complements.



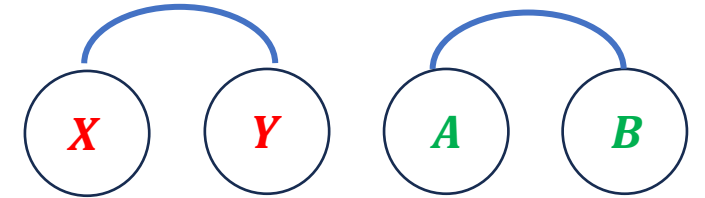
MMS-feasible ✓

Additive ✓



MMS-feasible ✓

Additive ✗



Not MMS-feasible ✗

Additive ✗

MMS-Feasible Valuations

Valuation is **MMS-feasible** if $\max\{v_i(X), v_i(Y)\} \geq \min\{v_i(A), v_i(B)\}$ for any $S \subseteq [m]$ and any two partitions $X \uplus Y = S$ and $A \uplus B = S$

Theorem (**Positive for Two Agents**):
PMMS exists for 1 monotone and 1 MMS-feasible valuation.

Theorem (**Negative for Three Agents**): There is an instance with 2 monotone and 1 MMS-feasible valuation that admits no **PMMS** allocation.

Open Problems:
Does **PMMS** exist for 1 monotone and 2 MMS-feasible?
Does **PMMS** exist for 3 MMS-feasible?

MMS-Feasible Valuations

MMS-feasible: $\max\{v_i(X), v_i(Y)\} \geq \min\{v_i(A), v_i(B)\}$.

Assumptions for 2 agents
2 monotone
1 monotone + 1 MMS-feasible
2 MMS-feasible

EFX	PMMS
exists	does not exist
exists	exists
exists	exists

Assumptions for 3 agents
3 monotone
2 monotone + 1 MMS-feasible
1 monotone + 2 MMS-feasible
3 MMS-feasible

EFX	PMMS
open	does not exist
exists	does not exist
exists	open
exists	open

Positive Results

Searching over instances requires assigning values to all bundles:
There are essentially $(2^m)!$ valuations.

We could restrict to valuations with few possible values:

Binary-valued (or **dichotomous**): $v_i(X) \in \{0, 1\}$.

There are 2^{2^m} such valuations.

We could restrict to valuations with few valuable bundles:

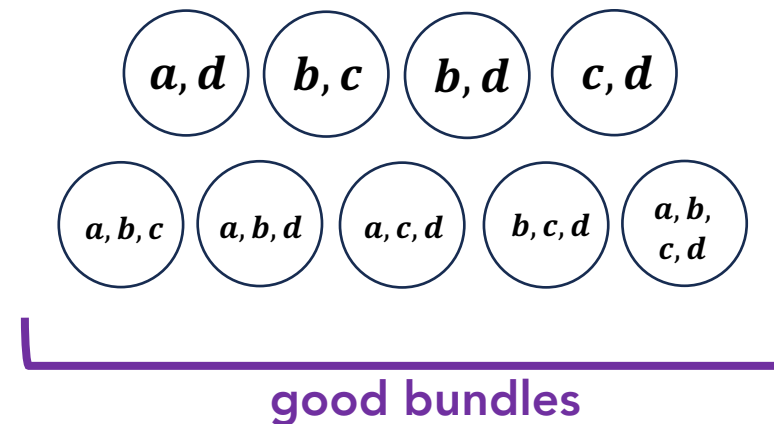
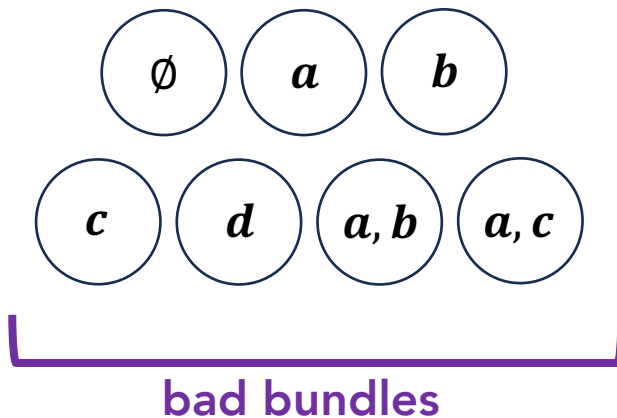
Pair-demand: $v_i(X) = \max_{T \subseteq X, |T| \leq 2} v_i(T)$.

There are $(m^2)!$ such valuations.

Positive Results

Theorem (**Positive**): **PMMS** exists for **binary-valued MMS-feasible** valuations.

Binary-valued (or **dichotomous**): $v_i(X) \in \{0, 1\}$.



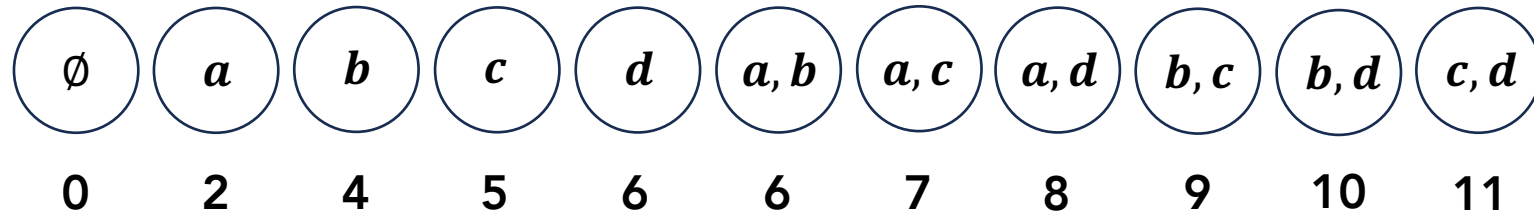
Does not exist for two **binary-valued** valuations without **MMS-feasibility**.

No monotonicity required! Applies to **goods**, **chores**, and **mixed manna**.

Positive Results

Theorem (Positive): PMMS exists for additive pair-demand valuations.

Additive pair-demand: $v_i(X) = \max_{T \subseteq X, |T| \leq 2} \sum_{g \in T} v_i(\{g\})$.



Does not exist for two general pair-demand valuations.

Open Problem: Does it exist for MMS-feasible pair-demand valuations?

Summary

Big Open Problems:

Does **EFX** exist for **additive valuations**?
Does **PMMS** exist for **additive valuations**?

We can make progress by showing
impossibility for **weaker** assumptions and
possibility for **stronger** assumptions.

PMMS is an **interesting** and **stronger** variant of **EFX**.

PMMS requires **new techniques**, possibly useful for **EFX**.