Online Fair Division with Subsidy: When Do Envy-Free Allocations Exist, and at What Cost?

Tomek Ponitka
Tel Aviv University

Joint work with

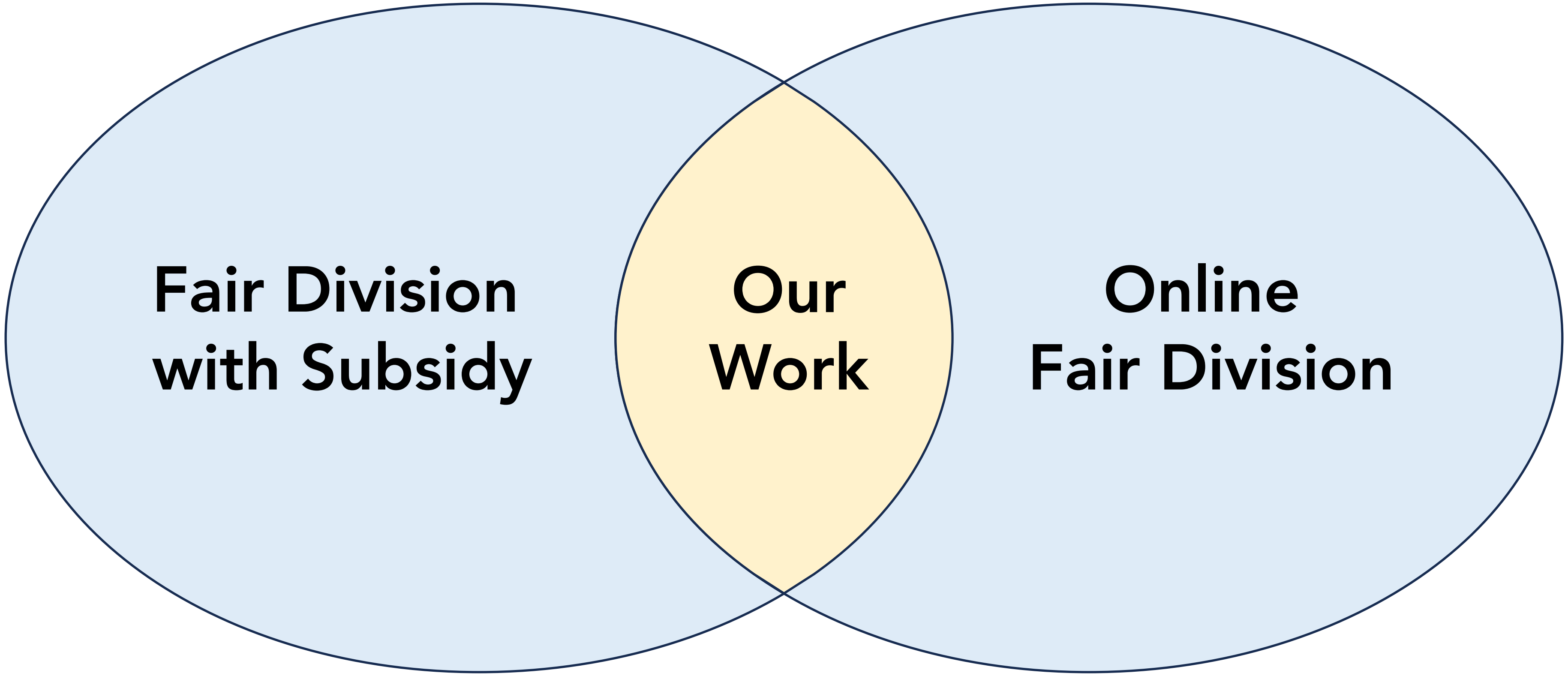


Pooja Kulkarni
Northwestern University

Ruta Mehta
University of Illinois Urbana-Champaign

Vishnu V. Narayan
IIT Bombay

Online Fair Division with Subsidy



Fair Division of Indivisible Goods

Fairly divide m indivisible goods among n agents.

	item a	item b	item c
agent 1	0.75	0.75	1
agent 2	1	1	0.5

For now, assume additive valuations,
where agent i's value for bundle X_i is $v_i(X_i) = \sum_{g \in X_i} v_{i,g}$

Envy-Freeness

Allocation is envy-free (EF) if $v_i(X_i) \geq v_i(X_j)$ for all i, j.

	item a	item b	item c
agent 1	0.75	0.75	1
agent 2	1	1	0.5

envy

$v_2(X_2) = 1$ $\quad$ $v_2(X_1) = 1.5$

Envy-free allocations might not exist.

Fair Division with Subsidy

Allocation X with subsidy vector $p = (p_1, \ldots, p_n)$ is envy-free if $v_i(X_i) + p_i \geq v_i(X_j) + p_j$ for all i, j.

	item a	item b	item c	subsidy
agent 1	0.75	0.75	1	0
agent 2	1	1	0.5	0.5

$v_2(X_2) + p_2 = 1.5$ $\quad$ $v_2(X_1) + p_1 = 1.5$

This allocation is envy-free with subsidy!

Fair Division with Subsidy

Allocation X with subsidy vector $p = (p_1, \dots, p_n)$
is envy-free if $v_i(X_i) + p_i \geq v_i(X_j) + p_j$ for all i, j.

Very active area of research.
Focus on minimizing total subsidy $\sum_{i \in [n]} p_i$.

[Maskin, 1987] [Tadenuma, Thomson, 1993] [Aragones, 1995] [Klijn, 2000]
[Halpern, Shah, SAGT 2019] [Brustle, Dippel, Narayan, Suzuki, Vetta, EC 2020]
[Caragiannis, Ioannidis, WINE 2021] [Barman, Krishna, Narahari, Sadhukhan, IJCAI 2022]
[Goko, Igarashi, Kawase, Makino, Sumita, Tamura, Yokoi, Yokoo, AAMAS 2022]
[Choo, Ling, Suksompong, Teh, Zhang, ORL 2024] [Wu, Zhang, Zhou, WINE 2023]
[Dupre la Tour, Fujii, 2025] [Klein Elmalem, Gonen, Segal-Halevi, AAMAS 2025]
[Aziz, Huang, Kimura, Saha, Sun, Suzuki, Yokoo, AAMAS 2025]
[Klein Elmalem, Gonen, Segal-Halevi, AAMAS 2026]

Fair Division with Subsidy

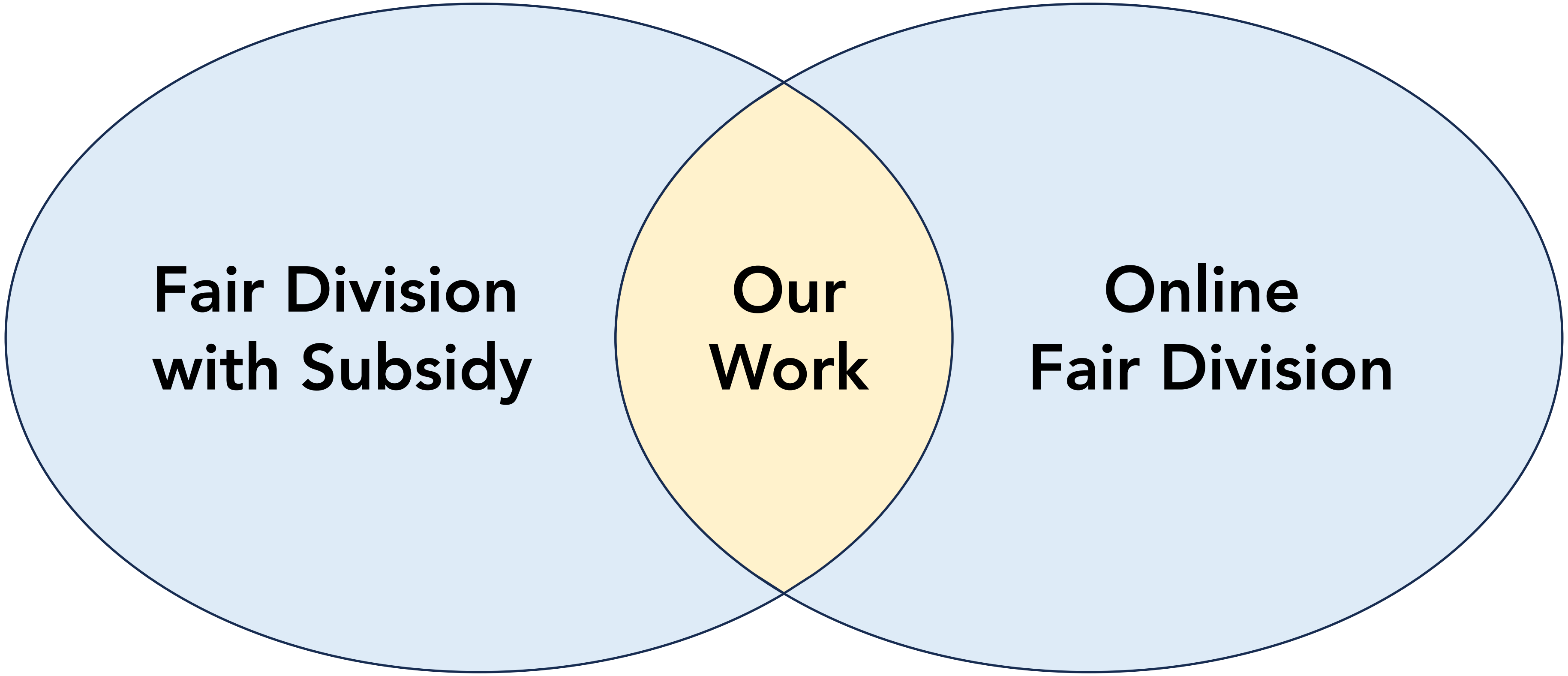
Allocation X with subsidy vector $p = (p_1, \dots, p_n)$
is envy-free if $v_i(X_i) + p_i \geq v_i(X_j) + p_j$ for all i, j.

When does X admit p making it envy-free?

Theorem: X is envy-freeable iff X is locally efficient:
there exists *no* permutation $\pi : [n] \to [n]$ with
$\sum_{i \in [n]} v_i\left(X_{\pi(i)}\right) > \sum_{i \in [n]} v_i(X_i)$.

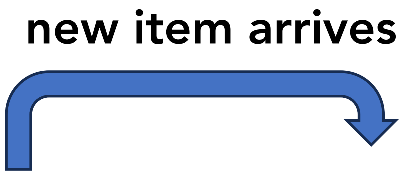
[Aragones, 1995] [Haake, Raith, Su, 2002] [Halpern, Shah, SAGT 2019]
[Hartline, Ieong, Mu'alem, Schapira, Zohar, ADT 2009]

Online Fair Division with Subsidy



Online Fair Division

Items are arriving online, while agents are offline.



	item a	item b	item c	item d
agent 1	0.75	0.75	1	1
agent 2	1	1	0.5	0.5

We can't modify past allocations.
Future items are unknown.

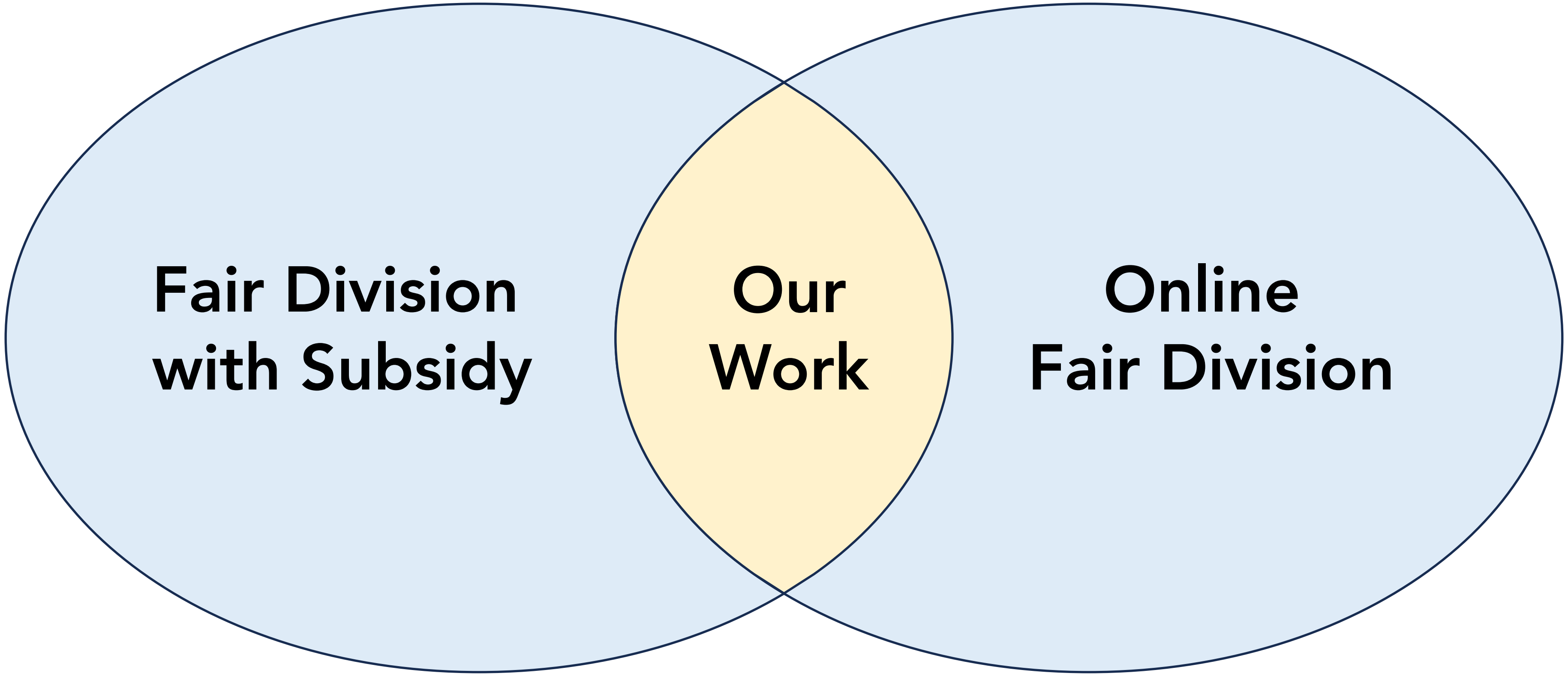
Online Fair Division

Items are arriving online, while agents are offline.

Very active area of research.
Strong impossibilities: e.g., cannot achieve envy-freeness up to some good (EF1).

[Aleksandrov, Aziz, Gaspers, Walsh, IJCAI 2015] [He, Procaccia, Psomas, Zeng, IJCAI 2019]
[Gkatzelis, Psomas, Tan, AAAI 2021] [Zhou, Bai, Wu, ICML 2023]
[Benade, Kazachkov, Procaccia, Psomas, Zeng, OR 2024] [Neoh, Peters, Teh, 2025]
[Halpern, Psomas, Verma, Xie, EC 2025] [Amanatidis, Lolos, Markakis, Turmel, SAGT 2025]
[Cookson, Ebadian, Shah, AAAI 2025] [Elkind, Lam, Latifian, Neoh, Teh, AAMAS 2025]
[Choo, Fu, Khu, Neoh, Poon, Teh, 2025] [Song, Tao, Wang, Zhang, 2025]
[Melissourgos, Protopapas, 2025] [Wang, Wei, AAAI 2026]

Online Fair Division with Subsidy



Starting Point

Theorem: For additive valuations, there is an online algorithm that is envy-free with total subsidy $\leq m(n-1)$.

Proof: Assign each item to agent who values it most.
maximizes welfare ⇒ local efficiency ⇒ envy-freeability

	item 1	⋯	item m	subsidy
agent 1	1	1	1	0
agent 2	$1-\epsilon$	$1-\epsilon$	$1-\epsilon$	$m(1-\epsilon)$
⋮	$1-\epsilon$	$1-\epsilon$	$1-\epsilon$	$m(1-\epsilon)$
agent n	$1-\epsilon$	$1-\epsilon$	$1-\epsilon$	$m(1-\epsilon)$

total subsidy $\geq m(n-1)$

Research Direction

Theorem: For additive valuations, there is an online algorithm that is envy-free with total subsidy $\leq m(n-1)$.

Question 1: Can we go beyond additive valuations?

Question 2: Is it possible to use small subsidy that does not grow with the number of items m?

Q1: Beyond Additive Valuations

	Can EF be achieved online? (Our Work)	Can EF be achieved offline? (Prior Work)
Additive	YES	YES
SPLC	YES	YES
Binary Submodular	NO	YES
Binary Supermodular	NO	YES
Budget Additive	NO	YES
General Monotone	NO	YES

Strong online vs. offline separation.
Three incomparable impossibilties for highly structured classes.

Q2: Small Subsidy

	Can we get EF with small subsidy?	(Mostly) tight online subsidy bounds	Offline subsidy bounds (Prior Work)
Additive	NO	$m(n-1)$	$n-1$
Rank-One	YES	$n(n+1)/2-1$	$n-1$
Restricted Additive	YES	$n(n-1)/2$	$n-1$
k-Demand	YES	$k(n-1)$	$n-1$
k-Valued	YES	$\leq n^2k^n$	$n-1$
Identical Monotone	YES	$n-1$	$n-1$

Offline subsidy for additive is $n-1$.
Online subsidy is $\Omega(n^2)$ even for structured subclasses of additive.

Q2: Small Subsidy

Thm: There is *no* online algorithm that achieves envy-freeness with total subsidy $< m(n-1)$ for additive valuations.

	item 1	item 2	item 3	$\cdots$	subsidy
agent 1	$1-\epsilon+2\delta$	$1-\epsilon+4\delta$	$1-\epsilon+8\delta$	$\cdots$	0
agent 2	$1-\epsilon+\delta$	$1-\epsilon+2\delta$	$1-\epsilon+4\delta$	$\cdots$	$\approx m$
$\vdots$	$1-\epsilon+\delta$	$1-\epsilon+2\delta$	$1-\epsilon+4\delta$	$\cdots$	$\approx m$
agent n	$1-\epsilon+\delta$	$1-\epsilon+2\delta$	$1-\epsilon+4\delta$	$\cdots$	$\approx m$

Local efficiency at every step forces all items to agent 1.

Connection to Envy Minimization

We focus on achieving envy-freeness with small subsidy.

Related line of work studies online envy minimization: maintain allocation with $v_i(X_j) - v_i(X_i) \leq E$ for small E.

[Aleksandrov, Aziz, Gaspers, Walsh, IJCAI 2015] [Benade, Kazachkov, Procaccia, Psomas, Zeng, OR 2024] [Halpern, Psomas, Verma, Xie, EC 2025] [Dang, Halpern, Makur, Psomas, Singh, Verma, 2026]

Lemma: Any online algorithm that maintains local efficiency and has envy bounded by E can yield envy-freeness with total subsidy $\leq E \cdot n(n-1)$.

Local efficiency is the source of hardness in our setting.

Summary

Question 1: Is it possible to maintain envy-freeability online beyond additive valuations?

Additive	YES
SPLC	YES

Binary Submodular	NO
Binary Supermodular	NO
Budget Additive	NO

Question 2: Is it possible to achieve envy-freeness online with small subsidy independent of m?

Additive	NO	$m(n-1)$

Rank-One	YES	$n(n+1)/2-1$
Restricted Additive	YES	$n(n-1)/2$
k-Demand	YES	$k(n-1)$
k-Valued	YES	$\leq n^2k^n$ (?)
Identical Monotone	YES	$n-1$