On Optimal Tradeoffs between EFX and Nash Welfare

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Setting

A resource allocation problem consists of

- a set of agents $[n] = \{1, \ldots, n\}$
- a set of indivisible goods $M = \{a, b, c, \ldots\}$
- a valuation function $v_i : 2^M \to \mathbb{R}^{\geq 0}$ for every agent i

Standard assumptions:

- v_i are normalized $\quad (v_i(\emptyset) = 0)$
- v_i are monotone $\quad (v_i(A) \leq v_i(B)$ for $A \subseteq B)$

We study two classes of valuation functions:

- additive $\quad (v_i(S) = \sum_{x \in S} v_i(x))$
- subadditive $\quad (v_i(A \cup B) \leq v_i(A) + v_i(B))$

The goal is to return an allocation $X = (X_1, \ldots, X_n)$

- $X_1, \ldots, X_n \subseteq M$ are disjoint subsets of goods
- X might be complete (i.e., $\bigcup_{i \in [n]} X_i = M$) or partial

Objective

Which of the following allocations should we choose?

	a	b	c
v_1	$6+\varepsilon$	3	1
v_2	$6+\varepsilon$	1	3

	a	b	c
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- Efficient:

$$\sum v_i(X_i) = 12 + \varepsilon$$

- Not fair:

$v_2(X_1) = 7 + \varepsilon$ and $v_2(X_2) = 3$

2 strongly envies 1

- Less efficient:

$$\sum v_i(X_i) = 10 + \varepsilon$$

- More fair:

$v_2(X_1) = 6 + \varepsilon$ and $v_2(X_2) = 4$

The envy is smaller

We are interested in the tradeoffs between efficiency (measured by Nash welfare) and fairness (captured by EFX).

Fairness

- An allocation is envy-free (EF) if $v_i(X_i) \geq v_i(X_j)$ for all i and j.
  EF is impossible to satisfy in general.
- An allocation is envy-free up to *any* good (EFX) if

  $$v_i(X_i) \geq v_i(X_j \setminus \{g\}) \text{ for } \textit{any } g \in X_j.$$

  The existence of EFX is an open problem (no proof for additive valuations, no counter-example for general monotone valuations!)
- An allocation is envy-free up to *one* good (EF1) if

  $$v_i(X_i) \geq v_i(X_j \setminus \{g\}) \text{ for } \textit{some } g \in X_j.$$

  EF1 exists for general monotone valuations. [LMMS'04]

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EF1 but not EFX

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EFX but not EF

Fairness

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  EF1 exists for general monotone valuations. [LMMS'04]
- An allocation is α-EFX for $\alpha \in [0, 1]$ if

$$v_i(X_i) \geq \alpha \cdot v_i(X_j \setminus \{g\}) \text{ for } \textit{any } g \in X_j.$$

  $\frac{1}{2}$-EFX exists for subadditive valuations. [PR'18]

  $(\varphi - 1 \approx 0.618)$-EFX exists for additive valuations. [AMN'20]

Efficiency

Social welfare measures:

- Utilitarian welfare: $\mathrm{UW}(X) = \sum_{1 \leq i \leq n} v_i(X_i)$
- Nash welfare: $\mathrm{NW}(X) = \prod_{1 \leq i \leq n} v_i(X_i)^{1/n}$

Why focus on Nash welfare?

- A maximum Nash welfare (MNW) allocation is more balanced relative to maximum utilitarian welfare allocations.

	a	b
v_1	2	2
v_2	1	1

Maximizes utilitarian welfare

	a	b
v_1	2	2
v_2	1	1

MNW allocation

- Nash welfare is *scale-free*.
- A MNW allocation is EF1. [CKMPSW'16]
- There is an instance where no EF1 allocation gets more than $O(1/\sqrt{n})$ fraction of maximum utilitarian welfare [BLMS'19] (i.e., the *price of fairness* of EF1 is $\Omega(\sqrt{n})$).

Nash welfare

Definition: $\mathrm{NW}(X) = \prod_{1 \leq i \leq n} v_i(X_i)^{1/n}$

We say that an allocation X is β-MNW for $\beta \in [0, 1]$ if

$$\mathrm{NW}(X) \geq \beta \cdot \text{maximum Nash welfare.}$$

	a	b	c
v_1	$6+\varepsilon$	3	1
v_2	$6+\varepsilon$	1	3

Nash welfare $= \sqrt{(9+\varepsilon) \cdot 3}$
This is a MNW allocation.

	a	b	c
v_1	$6+\varepsilon$	3	1
v_2	$6+\varepsilon$	1	3

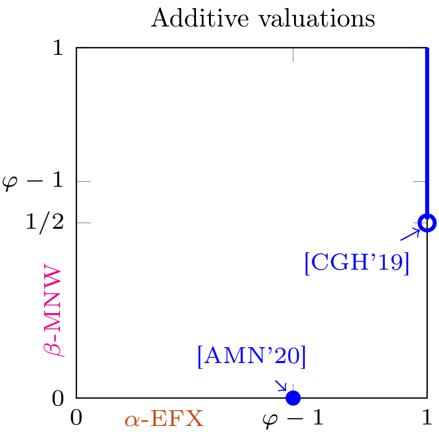
Nash welfare $= \sqrt{(6+\varepsilon) \cdot 4}$
This is a 0.94-MNW allocation.

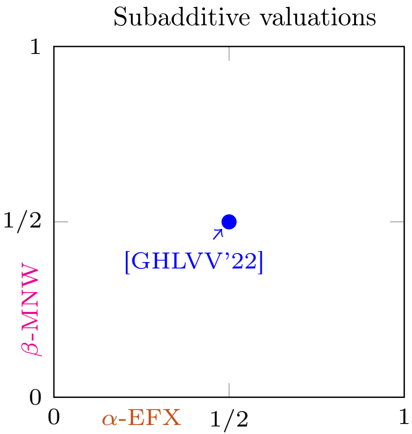
What is known about Nash welfare:

- Finding a MNW allocation is NP-hard
- Poly-time ($e^{1/e} \approx 1.45$)-approx. for additive valuations [BKV'18]
- Poly-time $(4+\varepsilon)$-approx. for submodular valuations [GHLVV'22]

Main results

Is there an α-EFX and β-MNW allocation (partial/complete)?





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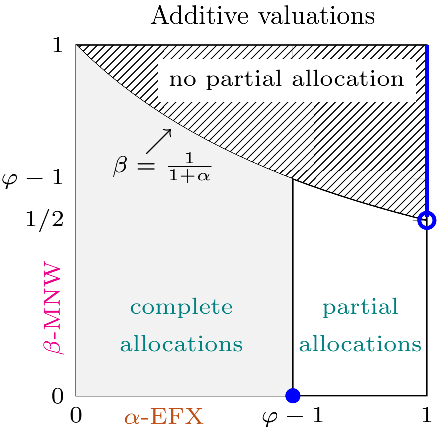

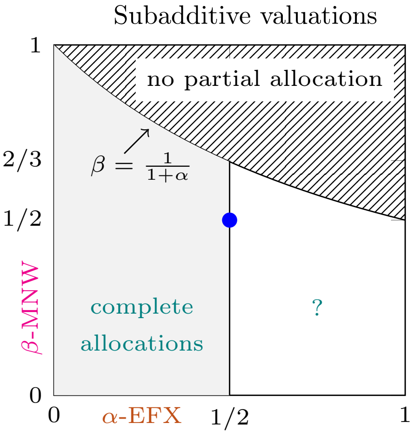

Note that $\varphi - 1 \approx 0.618$.

- We provide a new way to construct $(\varphi - 1)$-EFX for additive.
- We improve $\frac{1}{2}$-EFX, $\frac{1}{2}$-MNW to $\frac{1}{2}$-EFX, $\frac{2}{3}$-MNW for subadditive.

Proof for additive valuations ($\alpha = 1/2$)

Theorem: *Every instance with additive valuations admits a complete allocation that is $\frac{1}{2}$-EFX and $\frac{2}{3}$-MNW.*

Proof. We analyze a three-stage algorithm:

- Step 1. Start with a MNW allocation
- Step 2. Shrink some of the bundles to get a $\frac{1}{2}$-EFX allocation
- Step 3. Reallocate the removed items to get a complete allocation

Consider the running example:

	a	b	c
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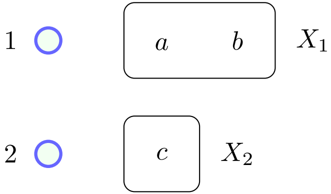

Step 1. We take the MNW allocation.

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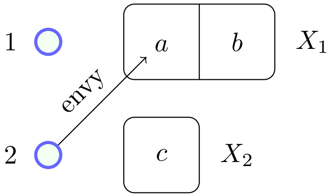

Step 2. The allocation is not $\frac{1}{2}$-EFX.

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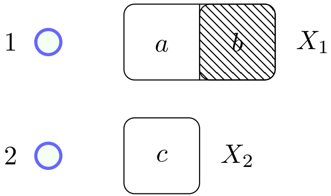

Step 2. Removing b from X_1 gives a partial, $\frac{1}{2}$-EFX, $\frac{2}{3}$-MNW alloc.

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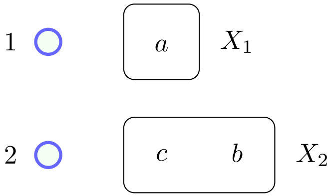
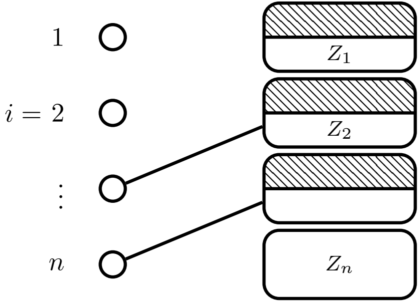


Step 3. Adding b to X_2 gives a complete, $\frac{1}{2}$-EFX, $\frac{2}{3}$-MNW alloc.

Proof for additive valuations ($\alpha = 1/2$)

Step 2. Shrink some of the bundles to get a $\frac{1}{2}$-EFX allocation

- Pick an unmatched agent i.
- If $v_i(Z_i) \geq (1/2) \cdot v_i(Z_j - g)$ for all j and g, then match i to Z_i.



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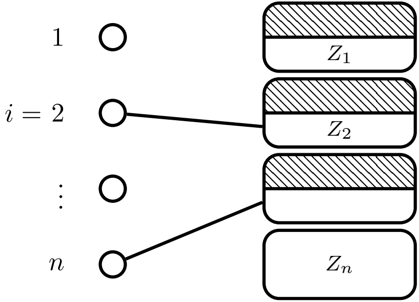

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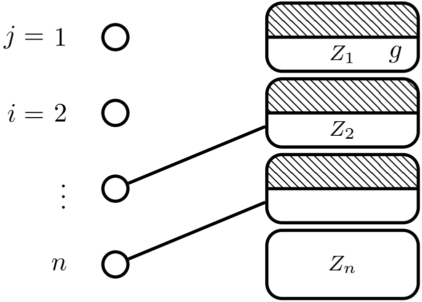
- Otherwise, pick j and g that maximize $v_i(Z_j - g)$, and then remove g from Z_j and match i to Z_j.



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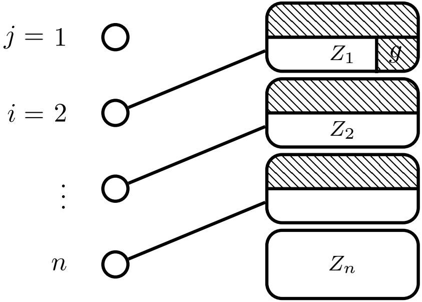

Note that $v_i(Z_j) > 2 \cdot v_i(Z_i)$

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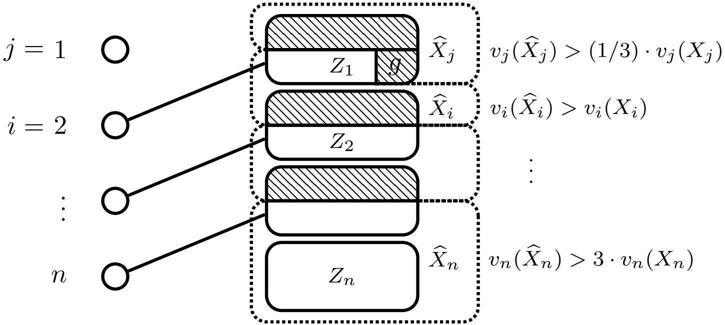


Claim: After every operation, we have $v_j(Z_j) \geq (2/3) \cdot v_j(X_j)$.

Proof. Suppose the contrary holds. We construct an allocation $\widehat{X}$ for which it holds that $\mathrm{NW}(\widehat{X}) > \mathrm{NW}(X)$ which gives a contradiction.

Proof for additive valuations ($\alpha = 1/2$)

Step 3. Reallocate the removed items to get a complete allocation

The *envy graph* contains an edge $(i \to j)$ if $v_i(X_j) > v_i(X_i)$.

- If there is an *envy cycle* of agents where each agent prefers the next agent's bundle, then reallocate the bundles along the cycle
- If there is an *unenvied* agent, give an unallocated item to her

Both operations preserve $\frac{2}{3}$-MNW.

Lemma: If we start step 3 with a α-EFX and γ-separated allocation, then at the end we obtain a $\min(\alpha, 1/(1+\gamma))$-EFX allocation.

An allocation is γ-separated for some $\gamma \in [0, 1]$ if

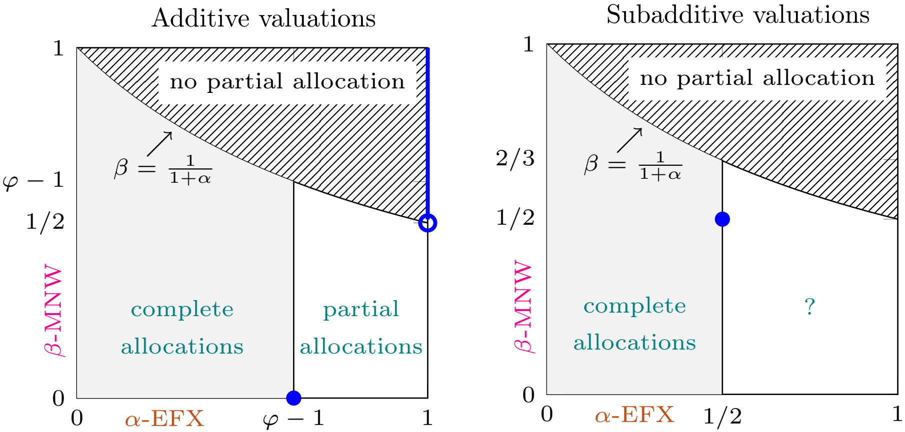
$$v_i(X_i) \geq (1/\gamma) \cdot v_i(g) \text{ for all unallocated } g.$$

To use the lemma, we show that step 2 guarantees α-separation.

Proof of Lemma. The first operation preserves $\frac{1}{2}$-EFX and $\frac{2}{3}$-MNW because the set of allocated bundles remains the same and every agent is weakly better off. For the second operation, observe that

$$v_j(X_i + g) = v_j(X_i) + v_j(g) \leq v_j(X_j) + \gamma \cdot v_j(X_j) = (1+\gamma) \cdot v_j(X_j).$$

Summary



- Fill in the remaining gaps.
- What about tradeoffs between EF1 and Nash welfare?
  For additive, EF1 and MNW is possible. [CKMPSW'16]
  For subadditive, $\frac{1}{4}$-EF1 and MNW is possible. [WLG'21]
  Is EF1 and β-MNW possible for subadditive?