On Optimal Tradeoffs between EFX and Nash Welfare

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May 11, 2023

Setting

A resource allocation problem consists of

- a set of agents $[n] = \{1, \ldots, n\}$
- a set of indivisible goods $M = \{a, b, c, \ldots\}$
- a valuation function $v_i : 2^M \to \mathbb{R}^{\geq 0}$ for every agent i

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The goal is to return an allocation $X = (X_1, \ldots, X_n)$

- $X_1, \ldots, X_n \subseteq M$ are disjoint subsets of goods
- X might be complete (i.e., $\bigcup_{i \in [n]} X_i = M$) or partial

Objective

Which of the following allocations should we choose?

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- Not fair:

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We are interested in the tradeoffs between efficiency (measured by Nash welfare) and fairness (captured by EFX).

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- An allocation is α-EFX for $\alpha \in [0, 1]$ if

$$v_i(X_i) \geq \alpha \cdot v_i(X_j \setminus \{g\}) \text{ for } \textit{any } g \in X_j.$$

  $\frac{1}{2}$-EFX exists for subadditive valuations. [PR'18]

  $(\varphi - 1 \approx 0.618)$-EFX exists for additive valuations. [AMN'20]

Efficiency

Social welfare measures:

- Utilitarian welfare: $\mathrm{UW}(X) = \sum_{1 \leq i \leq n} v_i(X_i)$
- Nash welfare: $\mathrm{NW}(X) = \prod_{1 \leq i \leq n} v_i(X_i)^{1/n}$

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- A maximum Nash welfare (MNW) allocation is more balanced relative to maximum utilitarian welfare allocations.

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MNW allocation

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- A MNW allocation is EF1. [CKMPSW'16]
- There is an instance where no EF1 allocation gets more than $O(1/\sqrt{n})$ fraction of maximum utilitarian welfare [BLMS'19] (i.e., the *price of fairness* of EF1 is $\Omega(\sqrt{n})$).

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What is known about Nash welfare:

- Finding a MNW allocation is NP-hard
- Poly-time ($e^{1/e} \approx 1.45$)-approx. for additive valuations [BKV'18]
- Poly-time ($4+\varepsilon$)-approx. for submodular valuations [GHLVV'22]

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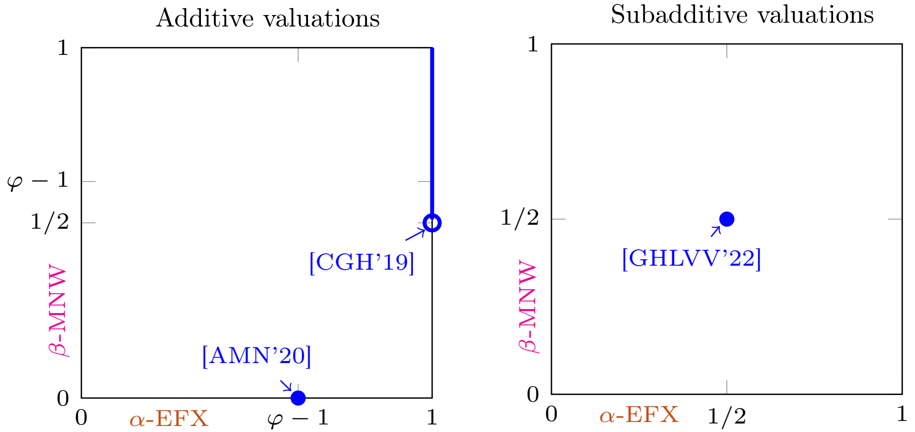

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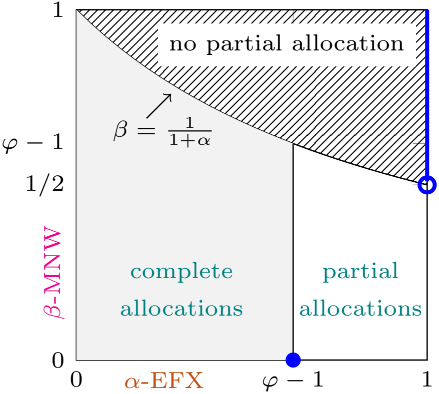
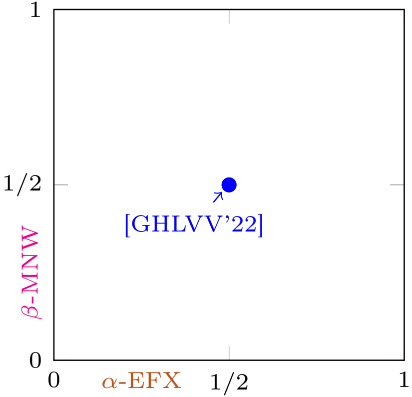


Subadditive valuations



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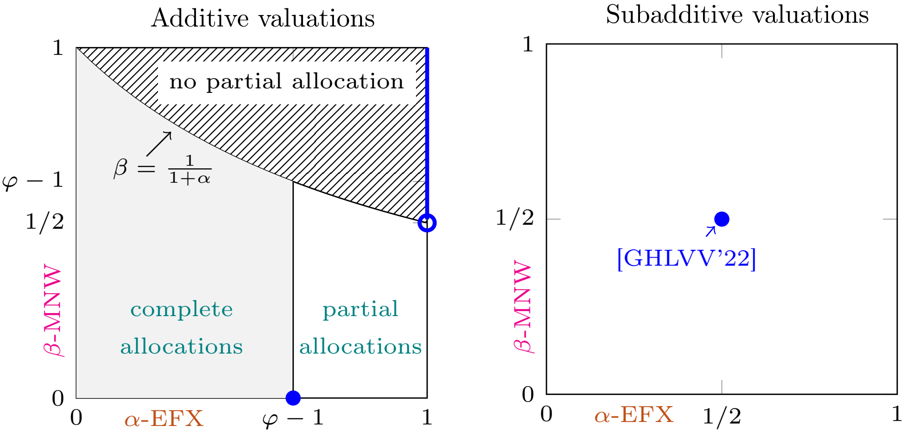

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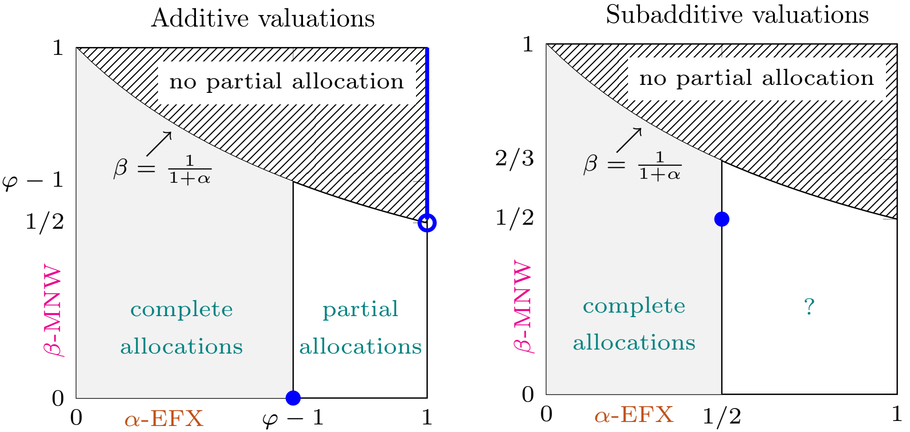

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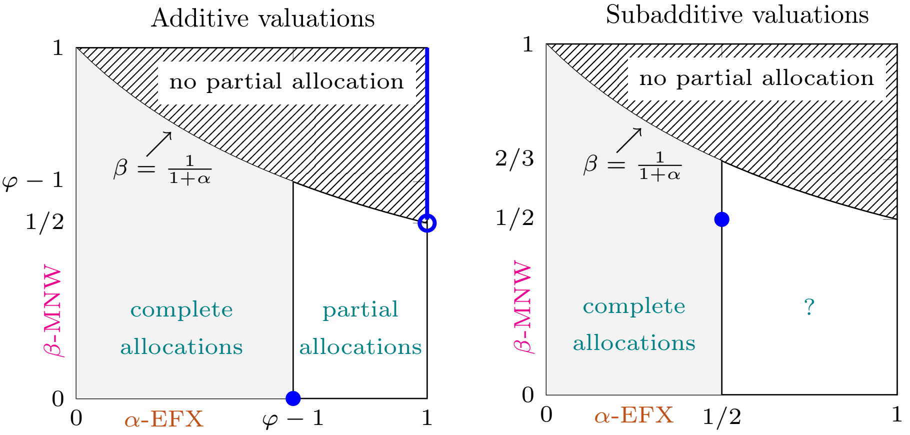

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- We improve $\frac{1}{2}$-EFX, $\frac{1}{2}$-MNW to $\frac{1}{2}$-EFX, $\frac{2}{3}$-MNW for subadditive.

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Theorem: *Every instance with additive valuations admits a complete allocation that is $\frac{1}{2}$-EFX and $\frac{2}{3}$-MNW.*

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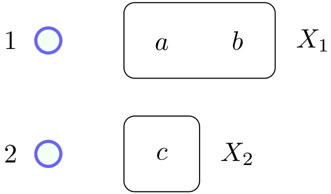

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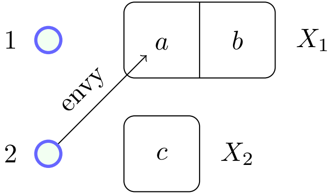

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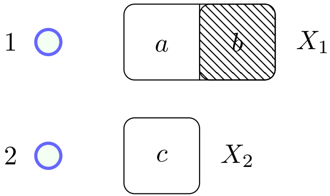

Step 2. Removing b from X_1 gives a partial, $\frac{1}{2}$-EFX, $\frac{2}{3}$-MNW alloc.

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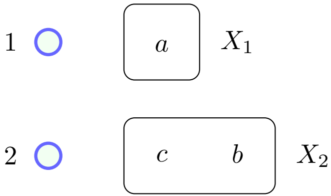

Step 3. Adding b to X_2 gives a complete, $\frac{1}{2}$-EFX, $\frac{2}{3}$-MNW alloc.

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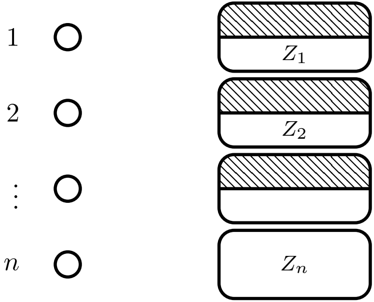

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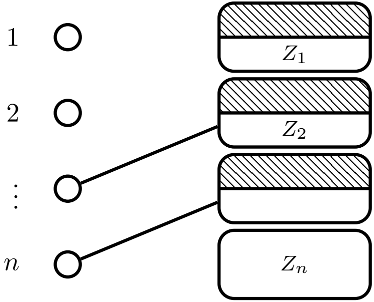

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Step 2. Shrink some of the bundles to get a $\frac{1}{2}$-EFX allocation

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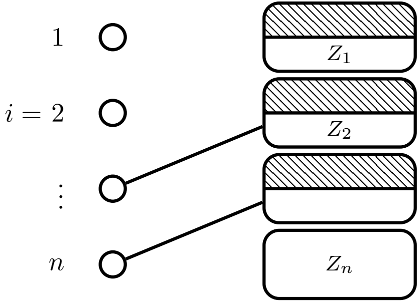

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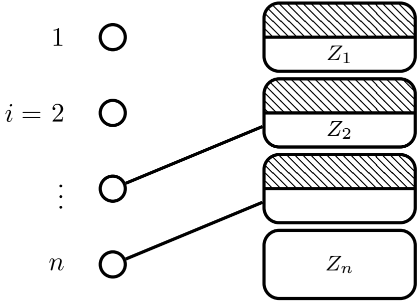

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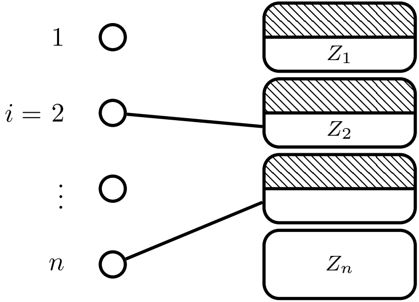

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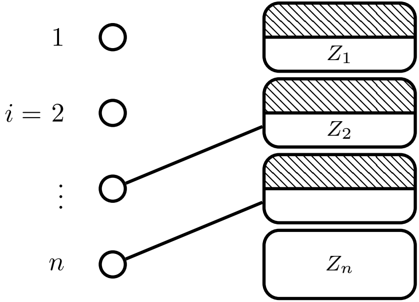

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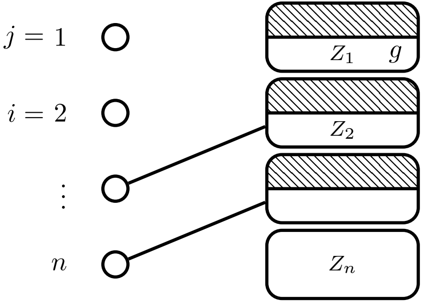

Proof for additive valuations ($\alpha = 1/2$)

Step 2. Shrink some of the bundles to get a $\frac{1}{2}$-EFX allocation

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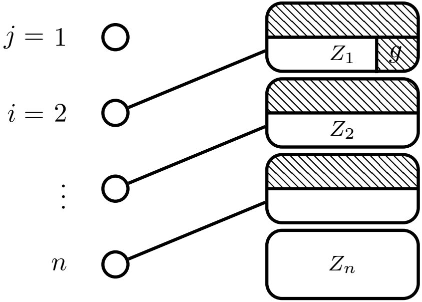

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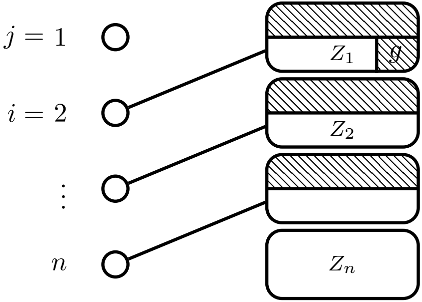

Note that $v_i(Z_j) > 2 \cdot v_i(Z_i)$

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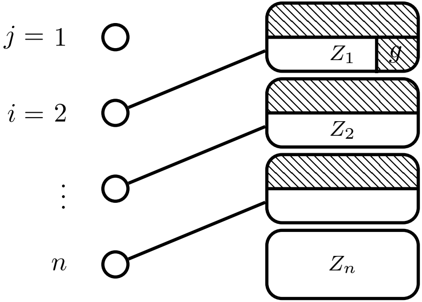

Claim: After every operation, we have $v_j(Z_j) \geq (2/3) \cdot v_j(X_j)$.

Proof. Suppose the contrary holds. We construct an allocation $\widehat{X}$ for which it holds that $\text{NW}(\widehat{X}) > \text{NW}(X)$ which gives a contradiction.

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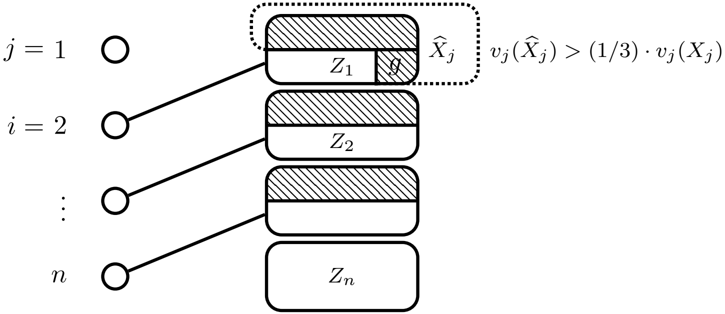

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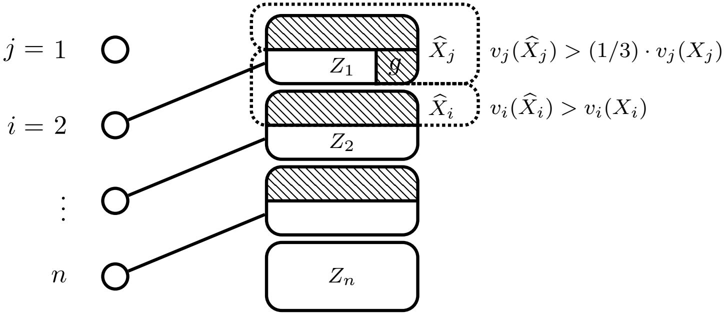

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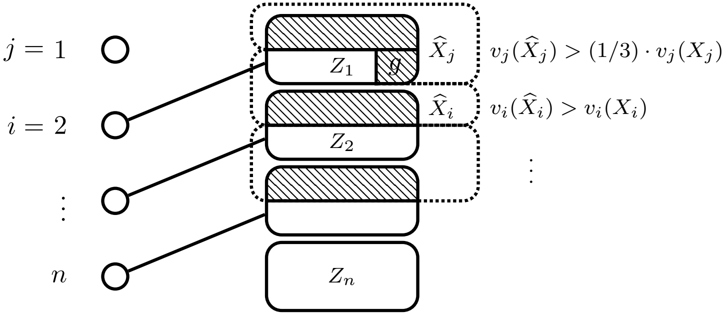

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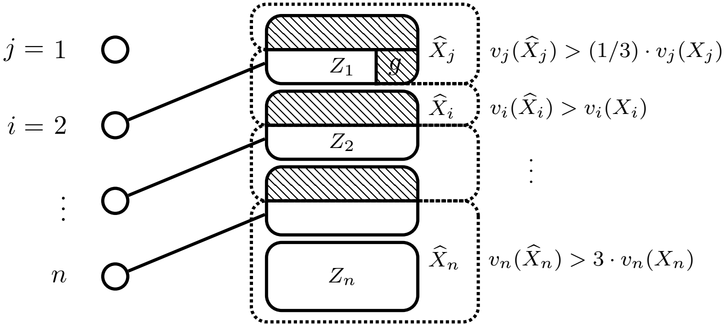

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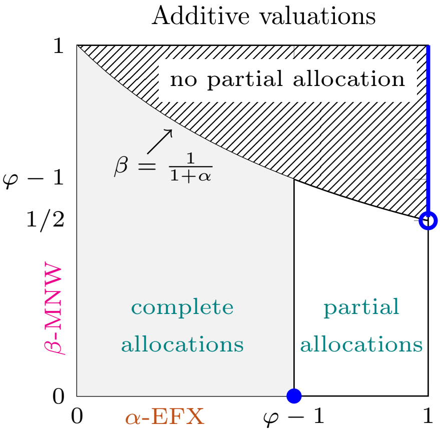
Proof of Lemma. The first operation preserves $\frac{1}{2}$-EFX and $\frac{2}{3}$-MNW because the set of allocated bundles remains the same and every agent is weakly better off. For the second operation, observe that

$$v_j(X_i + g) = v_j(X_i) + v_j(g) \leq v_j(X_j) + \gamma \cdot v_j(X_j) = (1 + \gamma) \cdot v_j(X_j).$$

Summary


Additive valuations
1
no partial allocation
$\beta = \frac{1}{1+\alpha}$
$\varphi - 1$
1/2
β-MNW
complete
allocations
partial
allocations
0
0
α-EFX
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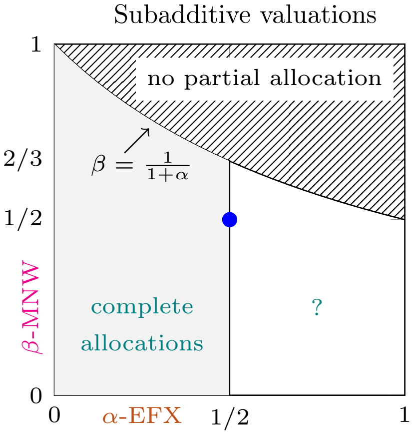
1


Subadditive valuations
1
no partial allocation
2/3
$\beta = \frac{1}{1+\alpha}$
1/2
β-MNW
complete
allocations
?
0
0
α-EFX
1/2
1

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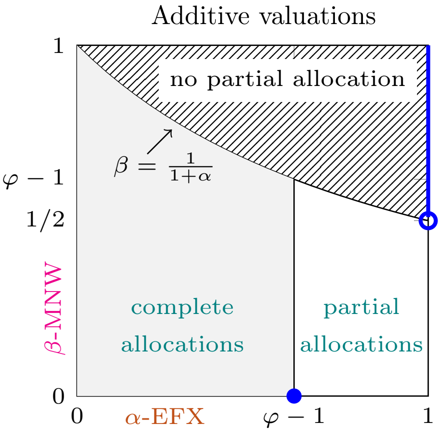

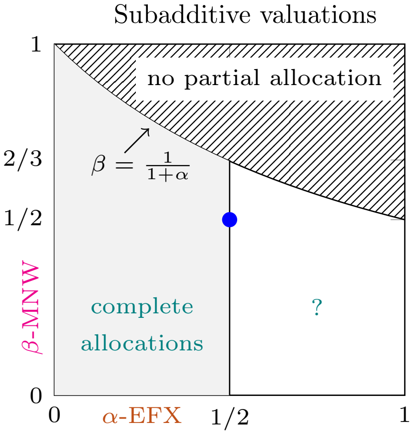

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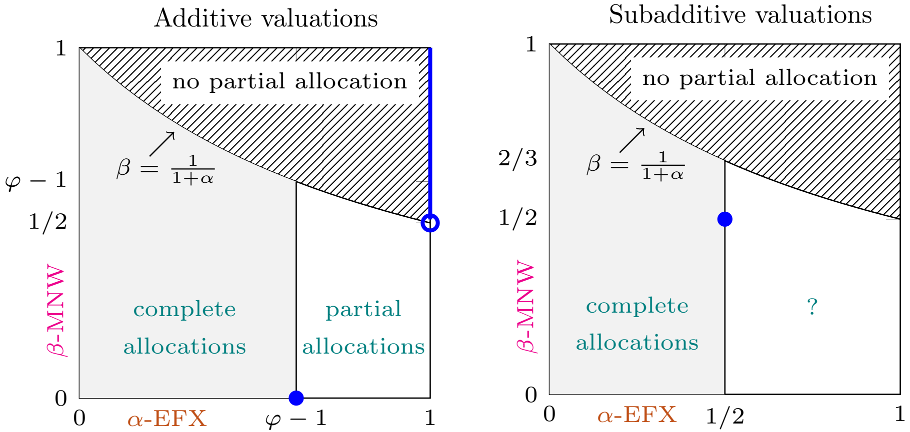

- Fill in the remaining gaps.
- What about tradeoffs between EF1 and Nash welfare?
  For additive, EF1 and MNW is possible. [CKMPSW'16]
  For subadditive, $\frac{1}{4}$-EF1 and MNW is possible. [WLG'21]
  Is EF1 and β-MNW possible for subadditive?