On Optimal Tradeoffs between EFX and Nash Welfare

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The Fair Division Problem

Allocate m indivisible goods among n agents in a fair manner.

Each agent i has a valuation function $v_i: 2^{[m]} \to \mathbb{R}^{\geq 0}$ over subsets of goods.

An allocation $(X_1, \dots, X_n)$ is complete if all the items are allocated, and partial otherwise.

Efficiency Notions

The Nash welfare is given by

$$\mathrm{NW}(X) = \prod v_i(X_i)^{1/n}.$$

An allocation is β-MNW for $\beta \in [0,1]$ if
$\mathrm{NW}(X) \geq \beta \cdot \text{maximum Nash welfare}.$

Fairness Notions

X is envy-free up to any good (EFX) if
$$\forall_{g \in X_j}\ v_i(X_i) \geq v_i(X_j - g).$$

Existence of EFX is an open problem. Hence, we consider approximations.

X is α-EFX for some $\alpha \in [0,1]$ if
$$\forall_{g \in X_j}\ v_i(X_i) \geq \alpha \cdot v_i(X_j - g).$$

The state-of-the-art approximations:

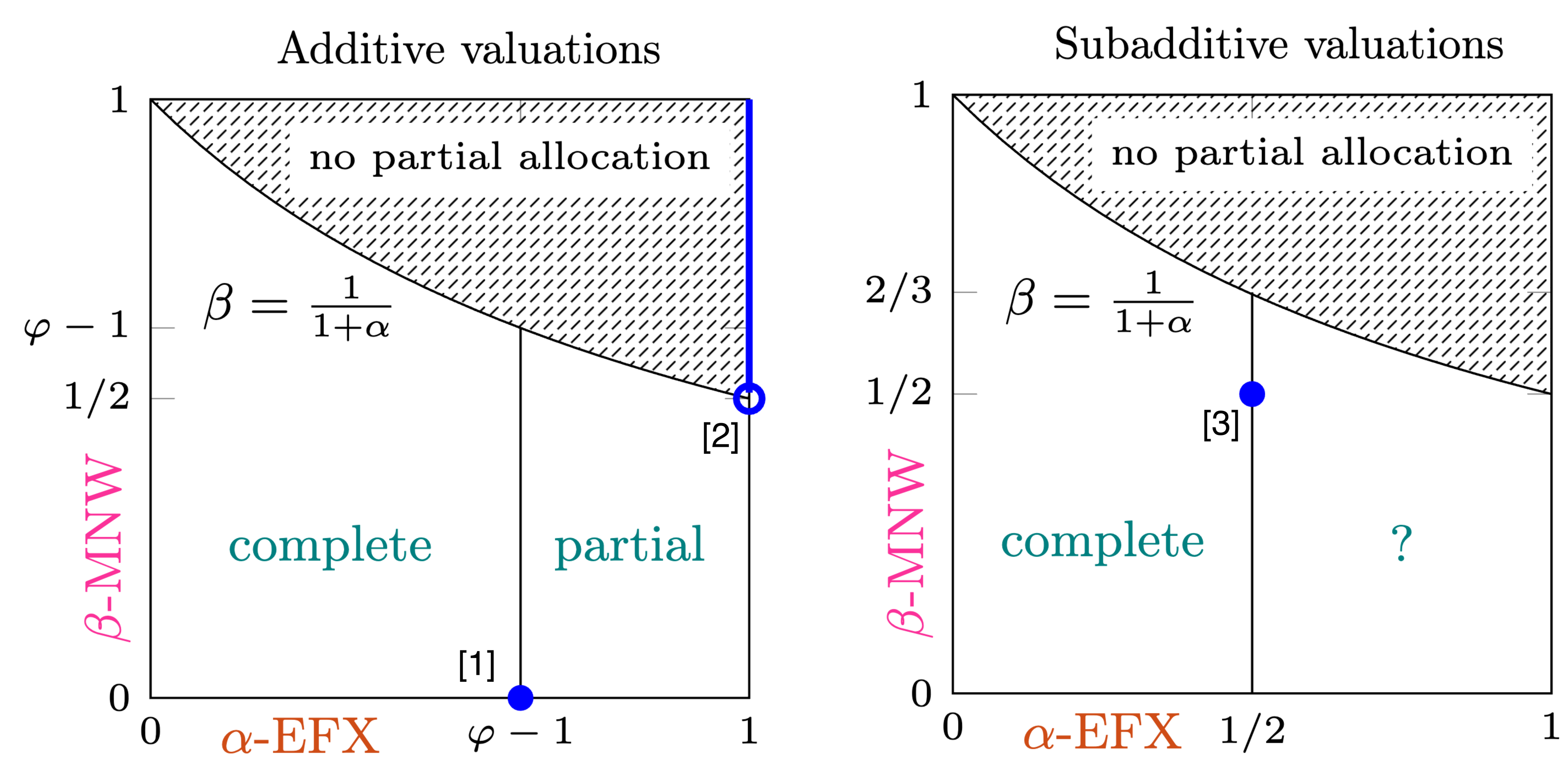
- $(\varphi - 1)$-EFX for additive valuations
- ½-EFX for subadditive valuations

Note that $(\varphi - 1) \approx 0.618$.

Main Results

Main Question: Is there an α-EFX and β-MNW allocation?

The positive (*complete* and *partial*) and negative (*no partial*) results are depicted in the following two figures.



Previous results:
[1] Amanatidis, Markakis, Ntokos, AAAI'20
[2] Caragiannis, Gravin, Huang, EC'19
[3] Garg, Husić, Li, Végh, Vondrák, STOC'23

Conclusions

The complete results match the state-of-the-art approximations of EFX.

- For additive, we show the existence of $(\varphi - 1)$-EFX and $(\varphi - 1)$-MNW. This improves the result of [AMN'20] who showed the existence of $(\varphi - 1)$-EFX with no efficiency guarantees.
- For subadditive, we show the existence of $^1/_2$-EFX and $^2/_3$-MNW. This improves the result of [GHLVV'23] who showed the existence of $^1/_2$-EFX and $^1/_2$-MNW.

The tradeoffs are tight due to an impossibility result.

The Allocation Construction

For additive, we use a simple three-step procedure:

Step 1. Take a maximum Nash welfare allocation.

Step 2. Keep removing elements from the envied bundles until the allocation becomes α-EFX.

Step 3. Reallocate the removed elements to make the allocation complete again.

For subadditive, we also need to include an operation in Step 2 which splits a bundle between two agents.

Making Allocations Complete (Step 3)

An allocation is γ-separated for some $\gamma \in [0,1]$ if
$v_i(X_i) \geq (1/\gamma) \cdot v_i(g)$ for any unallocated g.

Lemma. If we have with a partial allocation that is α-EFX and γ-separated, then we can add the unallocated goods to the allocation to obtain a $\min(\alpha, \frac{1}{1+\gamma})$-EFX allocation.

Proof. We use the well-known Envy Cycles procedure.

- If there is an unenvied agent i, give an unallocated good g to her. Observe that for any agent j,
  $v_j(X_i + g) \leq v_j(X_j) + \gamma \cdot v_j(X_j) = (1 + \gamma) \cdot v_j(X_j)$
  which gives the desired guarantee.
- Otherwise, there must be an envy cycle among the agents. We move the bundles along the cycle while preserving the EFX guarantees since the set of allocated bundles remains unchanged.

Remaining Analysis

It remains to show that at the end of Step 2, the allocation is α-separated (for additive) and $(\frac{1}{1+\alpha})$-MNW.