Breaking the Envy Cycle: Best-of-Both-Worlds Guarantees for Subadditive Valuations

Tomek Ponitka
Tel Aviv University

Joint work with

JOB MARKET

Michal Feldman
Tel Aviv University

Simon Mauras
INRIA

Vishnu V. Narayan
Tel Aviv University

Problem

Allocating limited resources via fair randomized lotteries

Affordable Housing

School Admissions

Residence Permits

Problem

Allocating limited resources via fair randomized lotteries

Our model:
agents with equal entitlement and diverse preferences over resources

Simple Model: Unit-Demand

Agents have unit-demand valuations:

$$v_i(X_i) = \max_{j \in X_i} v_{i,j}$$

X_i is agent i's allocation

A randomized lottery X is a probability distribution over integral allocations.

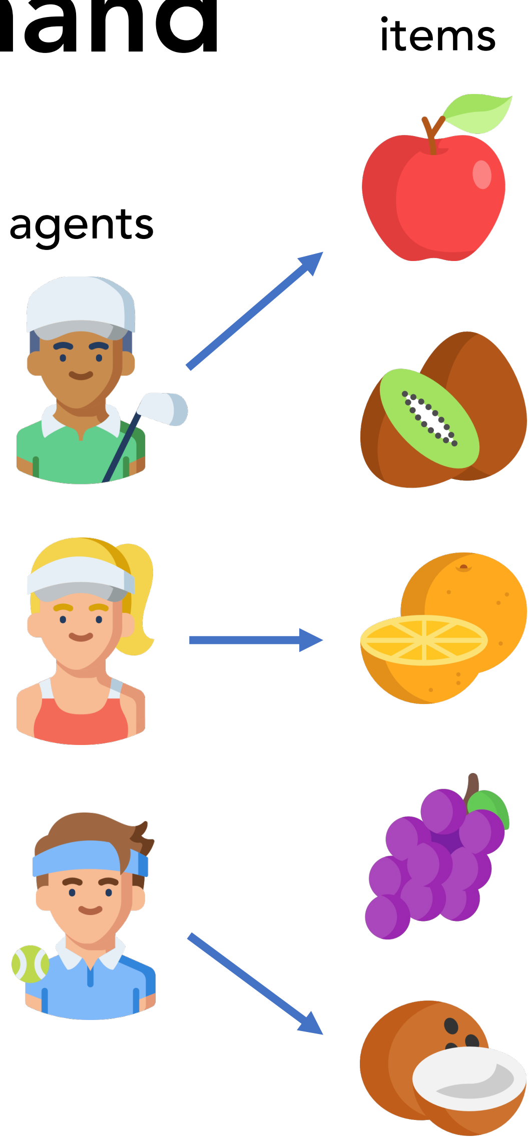
X is ex-ante envy-free (EF) if:

$$\mathbb{E}[v_i(X_i)] \geq \mathbb{E}[v_i(X_j)]$$

ex ante = before randomization

Ex-ante EF exists via simultaneous eating.

[Bogomolnaia, Moulin, 2001]



Intermediate Model: Additive

Agents have additive **valuations:**

$$v_i(X_i) = \sum_{j \in X_i} v_{i,j}$$

X_i is agent i's allocation

X **is ex-ante envy-free (EF) if** $\mathbb{E}[v_i(X_i)] \geq \mathbb{E}[v_i(X_j)]$.

ex-ante = before randomization

X **is ex-post EF1 if every realization satisfies:**

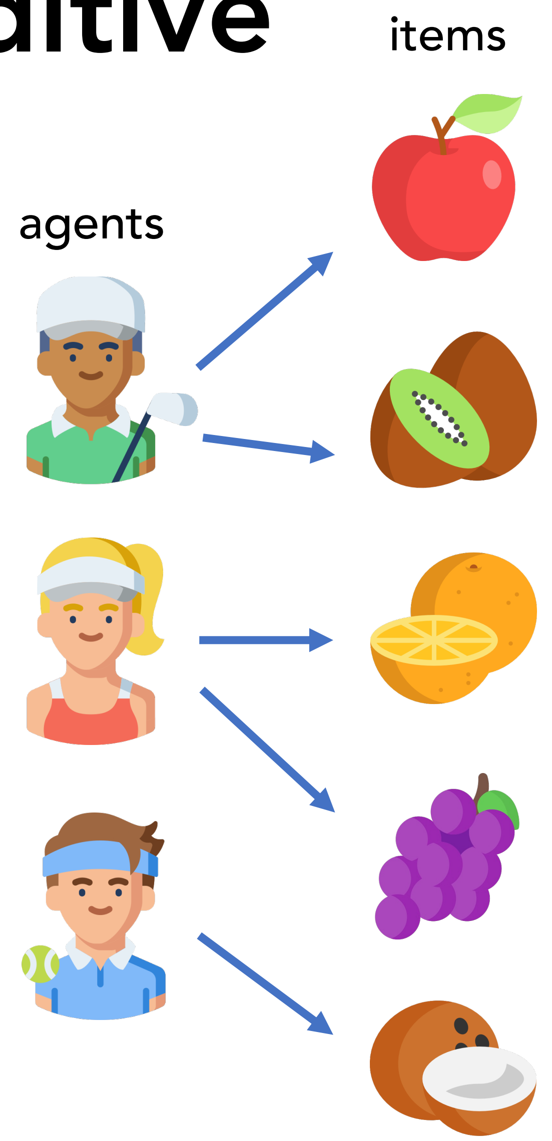
$$v_i(X_i) \geq v_i(X_j - g) \text{ for some } g \in X_i$$

ex-post = after randomization

Best-of-both-worlds guarantee:

Ex-ante EF and ex-post EF1 exists for additive vals.

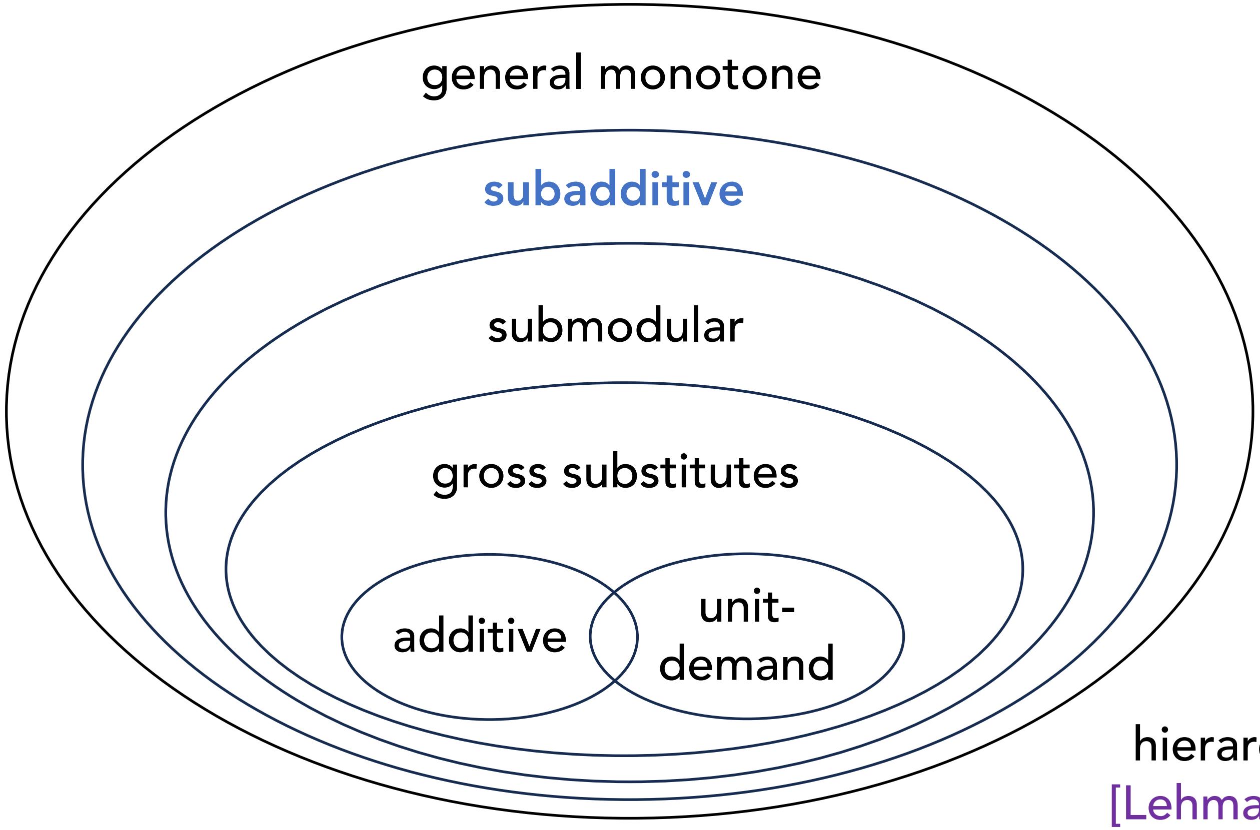
[Freeman, Shah, Vaish, 2020] and [Aziz, 2020]



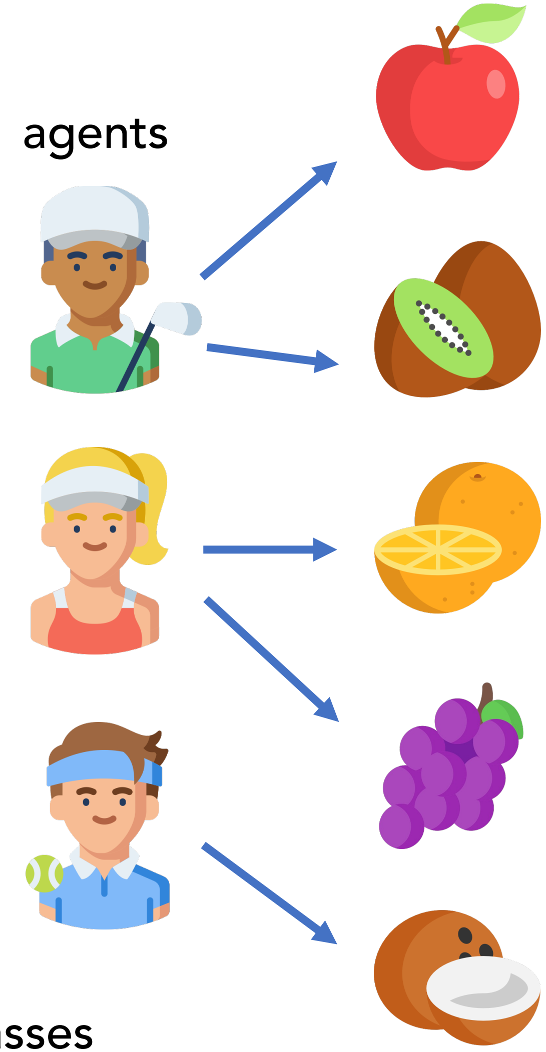
Combinatorial Model: Subadditive

Agents have subadditive valuations:

$$v_i(S \cup T) \leq v_i(S) + v_i(T)$$



hierarchy of valuation classes
[Lehman, Lehman, Nisan, 2001]



Research Direction

Starting point: [Freeman, Shah, Vaish, 2020] and [Aziz, 2020]

Ex-ante EF and ex-post EF1 exists for additive vals.

$(\mathbb{E}[v_i(X_i)] \geq \mathbb{E}[v_i(X_j)])$ $(v_i(X_i) \geq v_i(X_j - g)$ for some $g \in X_i)$ $(v_i(X_i) = \sum_{X_i} v_{i,j})$

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Q1: Can we get subadditive vals?

$(v_i(S \cup T) \leq v_i(S) + v_i(T))$

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Q2: Can we get ex-post EFX?

$(v_i(X_i) \geq v_i(X_j - g)$ for all $g \in X_i)$

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Q2: Can we get ex-post EFX?

$(v_i(X_i) \geq v_i(X_j - g)$ for all $g \in X_i)$

Theorem (impossibility result):

Ex-ante EF and ex-post EFX does not exist for subadditive vals.

Main Result

Theorem:
Ex-ante ½-EF and ex-post ½-EFX + EF1 exists for subadditive vals.

$\left(\mathbb{E}[v_i(X_i)] \geq 1/2 \cdot \mathbb{E}\left[v_i\left(X_j\right)\right]\right)$ $\left(v_i(X_i) \geq 1/2 \cdot v_i\left(X_j - g\right) \text{ for all } g \in X_i\right)$ $\left(v_i(S \cup T) \leq v_i(S) + v_i(T)\right)$

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- More general setting.
- Stronger ex-post guarantee.
- Weaker ex-ante guarantee.

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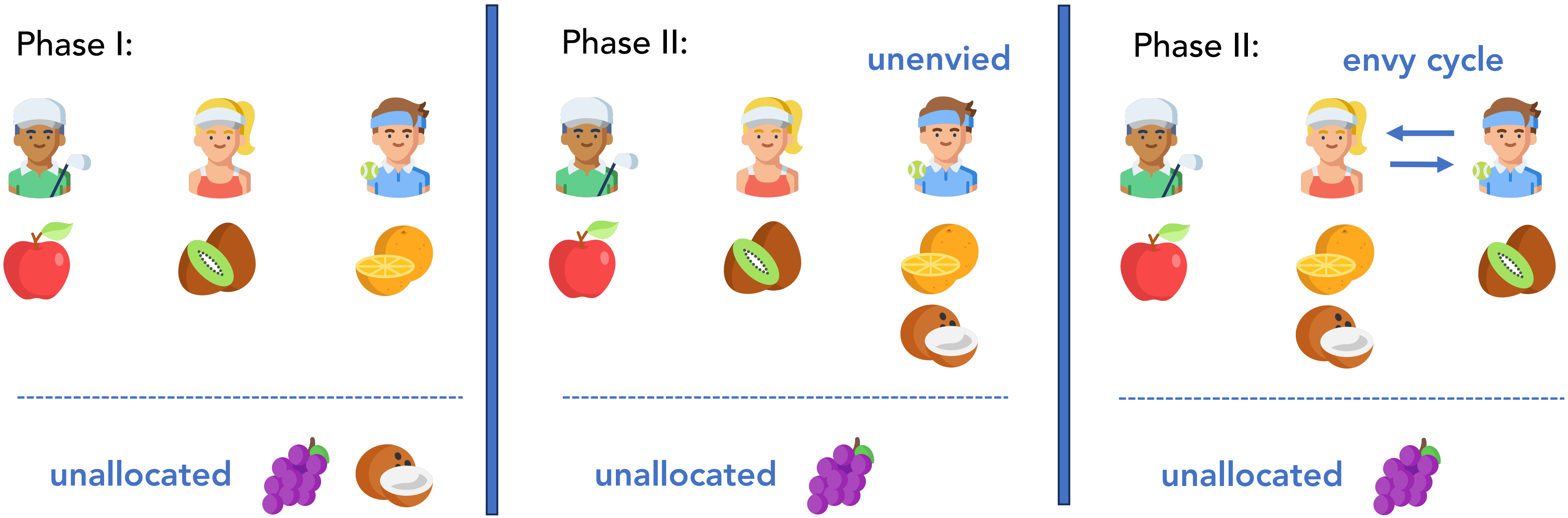
- Removing both approximation factors (½ and ½) is impossible.
- Going beyond subadditive to general monotone is impossible.

- The first best-of-both-worlds guarantee for subadditive valuations.
- ½-EFX is the best known approximation of EFX for subadditive.
- Proof via a careful randomization of the Envy Cycles procedure.

Breaking the Envy Cycle

Phase I: Allocate **one item** per agent. [Lipton, Markakis, Mossel, Saberi, 2004]

Phase II: In each iteration, give **unallocated item** to **unenvied agent**.
If all agents are envied, exchange bundles along **envy cycle**.



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[Lipton, Markakis, Mossel, Saberi, 2004] and [Plaut, Roughgarden, 2020]

For subadditive vals, under mild assumptions for Phase I,
the outcome of Envy Cycles always satisfies ex-post EF1 and ½-EFX.

$\left(v_i(X_i) \geq v_i(X_j - g) \text{ for some } g \in X_i\right)$ $\left(v_i(X_i) \geq 1/2 \cdot v_i(X_j - g) \text{ for all } g \in X_i\right)$

The main difficulty is to guarantee ex-ante ½-EF.

$\left(\mathbb{E}[v_i(X_i)] \geq 1/2 \cdot \mathbb{E}[v_i(X_j)]\right)$

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Guarantee ex-ante ½-EF by randomizing the choice of:

1. one item per agent
2. unallocated item
3. unenvied agent
4. envy cycle

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If all agents are envied, exchange bundles along envy cycle.

Guarantee ex-ante ½-EF by randomizing the choice of:

1. one item per agent ← simultaneous eating and Birkhoff rounding (see paper for details)
2. unallocated item ← arbitrary choices
3. unenvied agent ← arbitrary choices
4. envy cycle ← crucial part

Breaking the Envy Cycle

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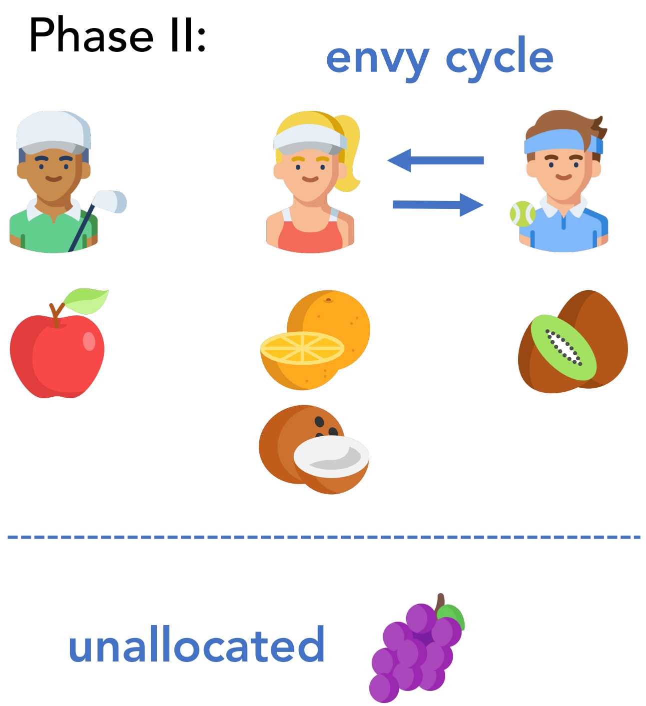


Which envy cycle to choose?

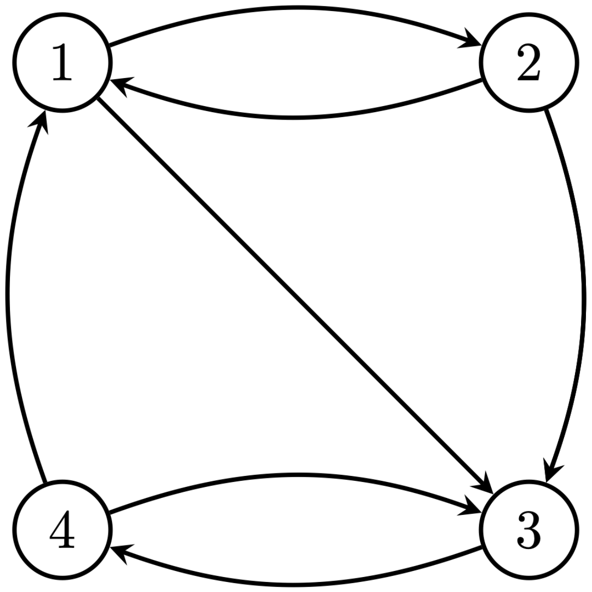
Next few slides:

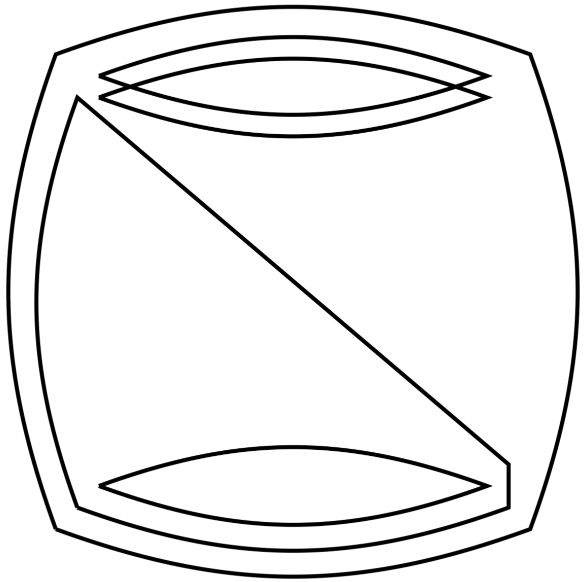
1. Define key property for envy cycle distribution
2. Intuition behind the key property
3. Construction satisfying the key property

Breaking the Envy Cycle

Envy graph: $(i \to j)$ if $v_i(X_i) < v_i(X_j)$

Distribution over envy cycles



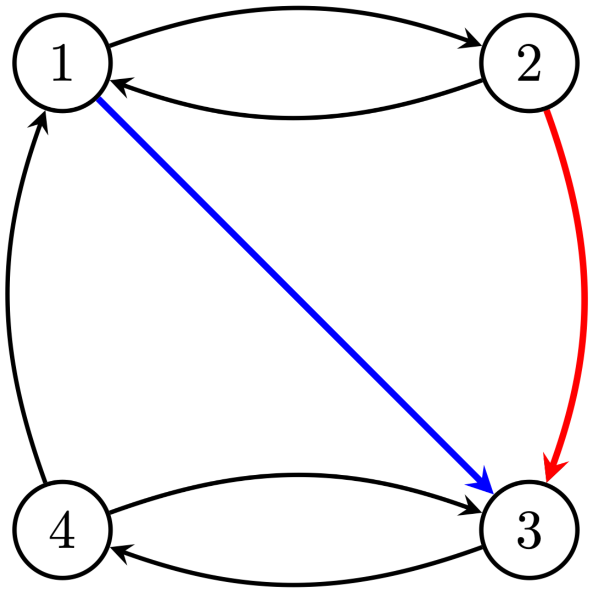


uniform distribution over 5 cycles

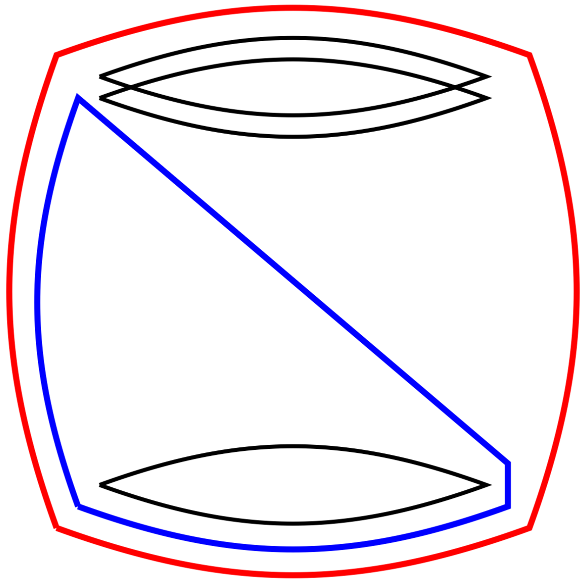
Breaking the Envy Cycle

Key property:
If agents i and j both envy k, they are equally likely to get X_k.

example: agents 1 and 2 both envy 3, and they get X_3 with probability 1/5 each.



Envy graph: $(i \to j)$ if $v_i(X_i) < v_i(X_j)$

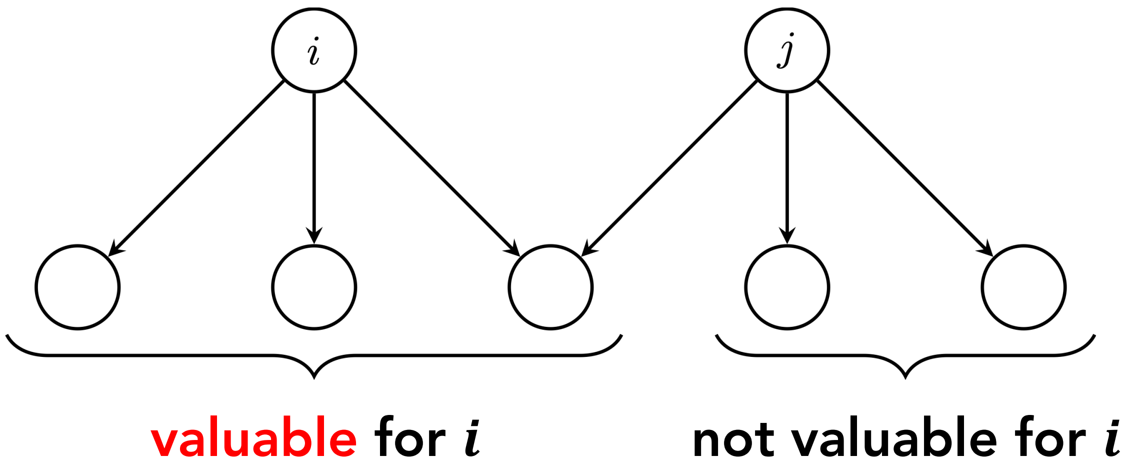


uniform distribution over 5 cycles

Breaking the Envy Cycle

Key property:
If agents i and j both envy k, they are equally likely to get X_k.

Intuition: An envy cycles distribution satisfying key property is envy-free.



i is at least as likely as j to get any of the valuable bundles

Breaking the Envy Cycle

Key property:

If agents i and j both envy k, they are equally likely to get X_k.

Stationary distribution of the **random walk** on the **transposed** envy graph

Breaking the Envy Cycle

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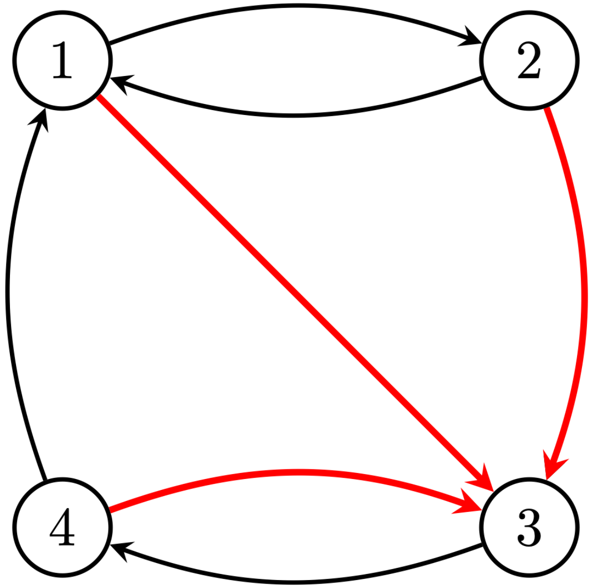
Stationary distribution of the random walk on the transposed envy graph



3 goes to 1, 2, 4 with probability 1/3 each

Breaking the Envy Cycle

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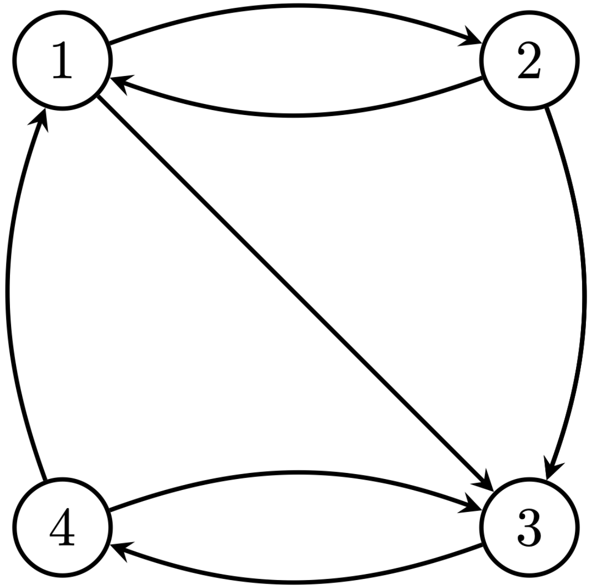
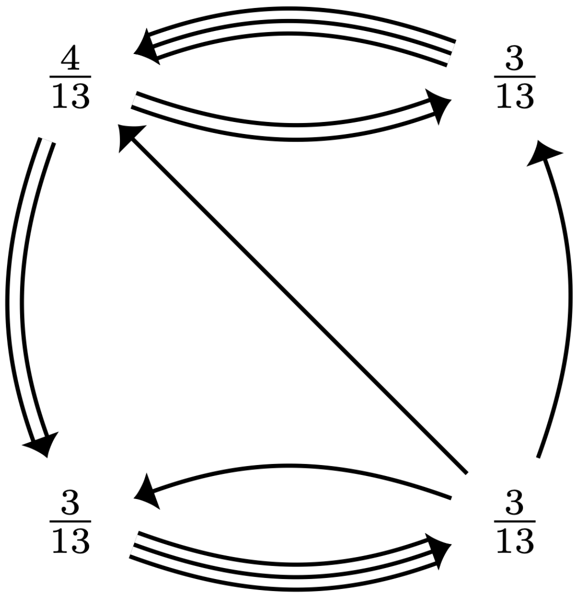
Stationary distribution of the **random walk** on the **transposed** envy graph





each edge = probability of 1/13

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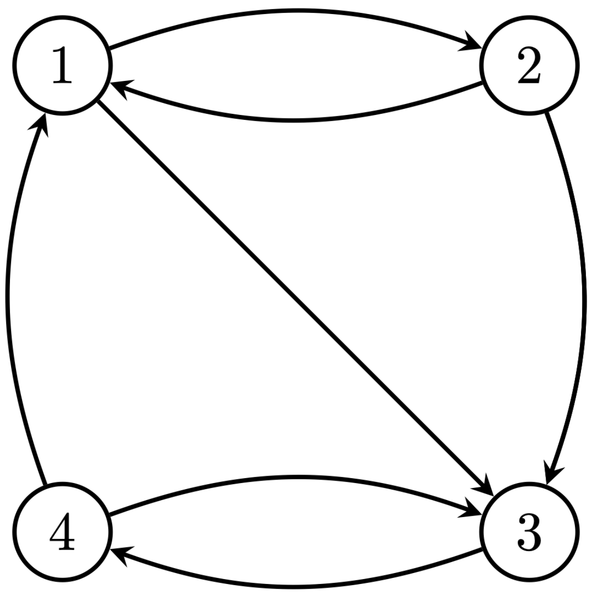

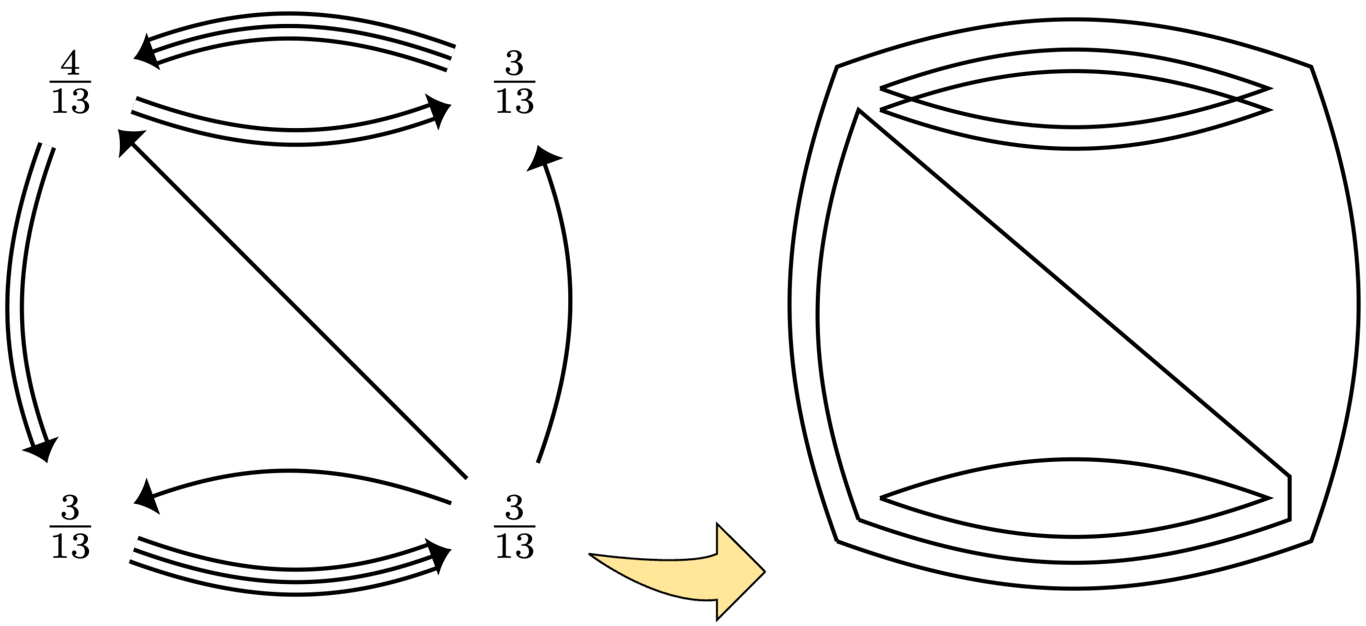

cycle decomposition by [MacQueen, 1981]

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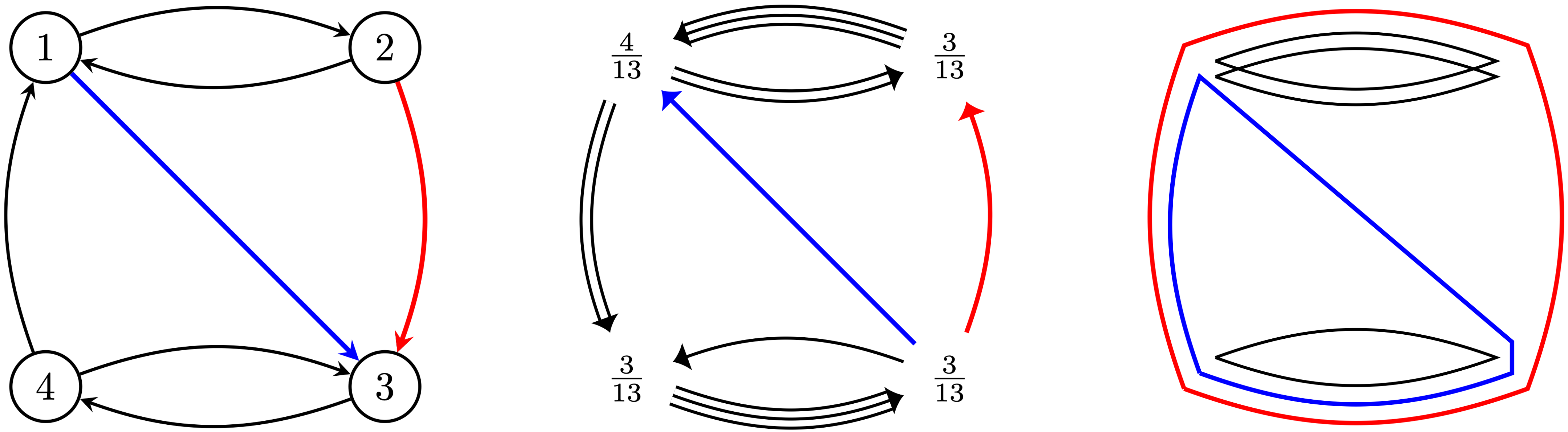
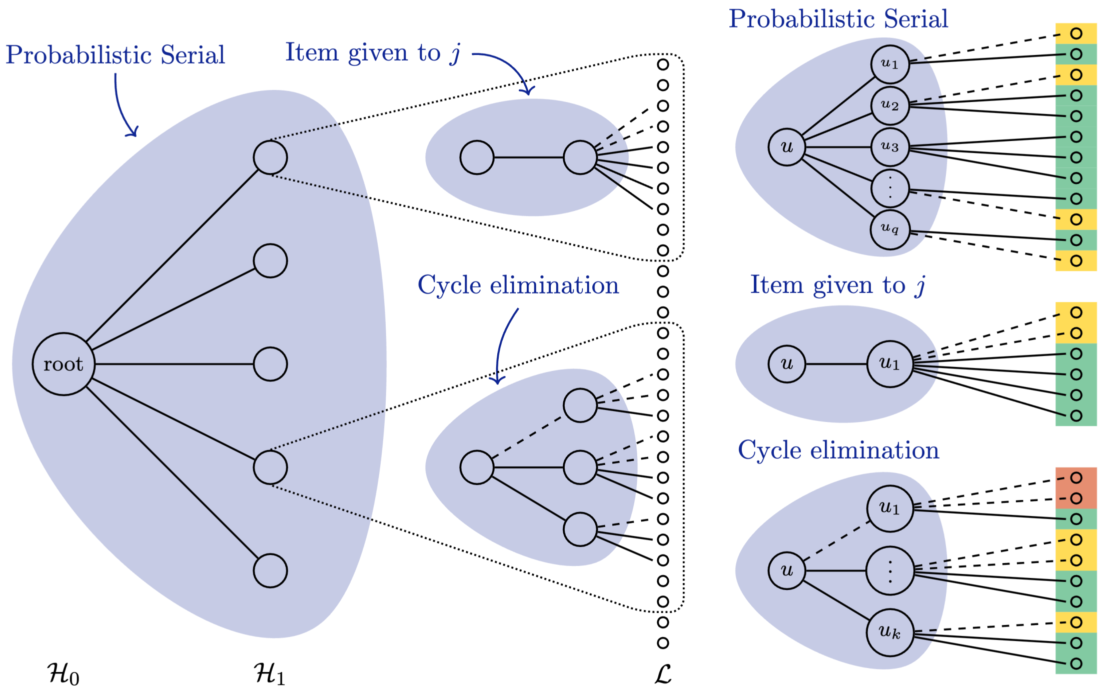


the key property holds because the random walk picks edges uniformly

Breaking the Envy Cycle

Key property:
If agents i and j both envy k, they are equally likely to get X_k.

See the paper for the proof that key property implies ex-ante ½-EF



Open Problems

Main result:
Ex-ante ½-EF and ex-post ½-EFX + EF1 exists for subadditive vals.

$(\mathbb{E}[v_i(X_i)] \geq 1/2 \cdot \mathbb{E}[v_i(X_j)])$ $(v_i(X_i) \geq 1/2 \cdot v_i(X_j - g)$ for all $g \in X_i)$ $(v_i(S \cup T) \leq v_i(S) + v_i(T))$

Open problem 1:
Does ex-ante EF and ex-post EF1 exist for subadditive vals?

Can we randomize Envy Cycles to get ex-ante EF?

Open problem 2:
Does ex-ante EF and ex-post EFX exist for additive vals?

Can we get a new impossibility result about EFX?

images: flaticon.com