Breaking the Envy Cycle: Best-of-Both-Worlds Guarantees for Subadditive Valuations

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The Fair Division Problem

Allocate m indivisible goods among n agents in a fair manner.

Each agent i has a valuation function $v_i\colon 2^{[m]} \to \mathbb{R}^{\geq 0}$ over subsets of goods.

An allocation is a collection of disjoint subsets of goods $(X_1, X_2, \dots, X_n)$.

Fairness Notions

Agent i envies agent j if $v_i(X_i) < v_i(X_j)$.

X is envy-free (EF) if
$$v_i(X_i) \geq v_i(X_j) \text{ for all } i, j.$$

Relaxations of EF:

X is EF up to one good (EF1) if
$$\exists_{g \in X_j}\ v_i(X_i) \geq v_i(X_j - g).$$

X is EF up to any good (EFX) if
$$\forall_{g \in X_j}\ v_i(X_i) \geq v_i(X_j - g).$$

EF for Randomized Allocations:

X is EF in expectation (ex-ante EF) if
$$\mathbb{E}[v_i(X_i)] \geq \mathbb{E}[v_i(X_j)].$$

Approximations of EF:

X is α-EFX for some $\alpha \in [0,1]$ if
$$\forall_{g \in X_j}\ v_i(X_i) \geq \alpha \cdot v_i(X_j - g).$$

X is ex-ante α-EF for some $\alpha \in [0,1]$ if
$$\mathbb{E}[v_i(X_i)] \geq \alpha \cdot \mathbb{E}[v_i(X_j)].$$

Main Contributions

Envy-freeness is impossible to satisfy in general.
There are two ways to deal with this.

Randomization
Ex-ante EF is possible.

Relaxations
EF1 and ½-EFX are known to exist.

Can we do both **simultaneously**?
This is called a best-of-both-worlds guarantee.

Theorem (Freeman et al.'20): For every instance with additive valuations, there is a distribution over EF1 allocations that is ex-ante EF.

This works only for additive valuations. Can we extend this to subadditive?

This gives only EF1 but not EFX. Can we get EFX?

We cannot get both improvements simultenously.

Impossibility: There is an instance with submodular valuations where no distribution over EFX allocations is ex-ante EF.

Therefore, we need to use approximation. We obtain the following result.

Main result: For every instance with subadditive valuations, there is a distribution over EF1 and ½-EFX allocations that is ex-ante ½-EF.

Conclusions

1. This is the first best-of-both-worlds result for subadditive valuations.
2. ½-EFX is the best-known approximation of EFX for subadditive valuations.
3. Gives a new technique to randomize the Envy Cycles procedure, which has seen widespread use in a large variety of fair division applications.

The Envy Cycles Procedure

Phase I: Allocate one item per agent, satisfying weak separation, i.e. $v_i(X_i) \geq v_i(g)$ for every unassigned item g.

Phase II: In each iteration, give a remaining item to an unenvied agent. If no agent is currently unenvied, reallocate the (fixed) bundles along an envy cyle.

Theorem (Plaut and Roughgarden'18): On termination, the resulting allocation is EF1 and ½-EFX.

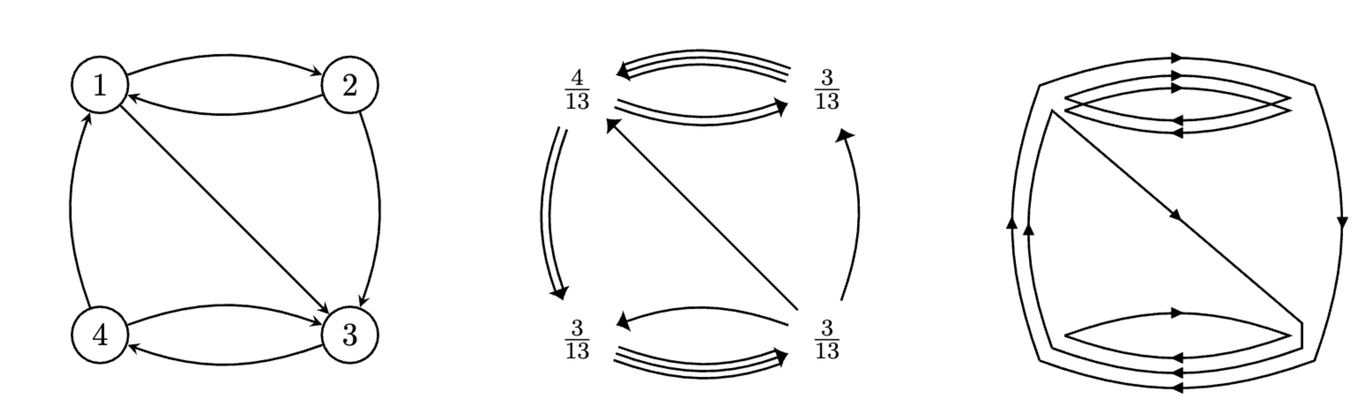
Randomizing Envy Cycles for Ex-Ante Fairness

Randomizing Phase I: Here we require a stronger property (strong separation): for a random allocation X, for every item g that has positive probability of being unassigned, $v_i(X_i) \geq v_i(g)$ in any outcome of the randomization.

We use Probabilistic Serial with Birkhoff rounding (Bogomolnaia and Moulin'01) that satisfies strong separation.

Randomizing Phase II: We present a new technique to randomize over the available envy cycles in each step, inspired by the work of MacQueen'81 on circuit processes. Put simply, our key lemma ensures that during cycle elimination, any pair of agents that both envy some bundle have equal probability of receiving that bundle.

Key Lemma. There is a probability distribution $(c_t, p_t)_t$ over the cycles of the envy graph s.t. for any arcs (i_1, j) and (i_2, j),
$$\textstyle\sum_{t:(i_1,j)\in c_t} p_t = \sum_{t:(i_2,j)\in c_t} p_t\,.$$



(i) Strongly connected graph (ii) Stationary distribution (iii) Cycle distribution